New Ranks for Even-Order Tensors and Their Applications in Low-Rank Tensor Optimization

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Abstract

In this paper, we propose three new notions of (even-order) tensor ranks, to be called the M-rank, the symmetric M-rank, and the strongly symmetric M-rank. We discuss the bounds between these new tensor ranks and the CP(CANDECOMP/PARAFAC)-rank and the symmetric CP-rank of an even-order tensor. In particular, we show: (1) these newly defined ranks actually coincide with each other if the even-order tensor in question is super-symmetric; (2) the CP-rank and symmetric CP-rank for a fourth-order tensor can be both lower and upper bounded (up to a constant factor) by the corresponding M-rank, and the bounds are tight for asymmetric tensors. In addition, we manage to identify a class of tensors whose CP-rank can be easily determined, and the CP-rank of the tensor equals the M-rank but is strictly larger than the Tucker rank. Since the M-rank is much easier to compute than the CP-rank, we propose to replace the CP-rank by the M-rank in the low-CP-rank tensor recovery model. Numerical results suggest that when the CP-rank is not very small our method outperforms the low-n-rank approach which is a currently popular model in low-rank tensor recovery. This shows that the M-rank is indeed an effective and easily computable approximation of the CP-rank.

Keywords: Matrix Unfolding, Tensor Decomposition, CP-Rank, Tensor Completion, Robust Tensor Recovery.

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1 Introduction

Tensor data have appeared frequently in applications such as computer vision [42], psychometrics [15, 8], diffusion magnetic resonance imaging [13, 3, 36], quantum entanglement problem [17], spectral hypergraph theory [18] and higher-order Markov chains [29]. Tensor-based multi-dimensional data analysis has shown that tensor models can take full advantage of the multi-dimensionality structures of the data, and generate more useful information. A common observation for huge-scale data analysis is that the data exhibits a low-dimensionality property, or its most representative part lies in a low-dimensional subspace. To take advantage of this low-dimensionality feature of the tensor data, it becomes imperative that the rank of a tensor, which is unfortunately a notoriously thorny issue, is well understood and computationally manageable. The most commonly used definition of tensor rank is the so-called *CP-rank*, where "C" stands for CANDECOMP while "P" corresponds to PARAFAC and these are two alternate names for the same tensor decomposition. The CP-rank to be introduced below, is the most natural notion of tensor rank, which is also very difficult to compute numerically.

Definition 1.1 *Given a d-th order tensor $\mathcal{F} \in \mathbb{C}^{n_1 \times n_2 \times \cdots \times n_d}$ in complex domain, its CP-rank (denoted as $\mathrm{rank}_{CP}(\mathcal{F})$) is the smallest integer r exhibiting the following decomposition*

$$\mathcal{F} = \sum_{i=1}^{r} a^{1,i} \otimes a^{2,i} \otimes \cdots \otimes a^{d,i}, \tag{1}$$

where $a^{k,i} \in \mathbb{C}^{n_i}$ for $k = 1, \ldots, d$ and $i = 1, \ldots, r$. Similarly, for a real-valued tensor $\mathcal{F} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}$, its CP-rank in the real domain (denoted as $\mathrm{rank}^{\mathbb{R}}_{CP}(\mathcal{F})$) is the smallest integer r such that there exists a real-valued decomposition (1) with $a^{k,i} \in \mathbb{R}^{n_i}$ for $k = 1, \ldots, d$ and $i = 1, \ldots, r$.

An extreme case is $r = 1$, where $\mathcal{F}$ is called a rank-1 tensor in this case. For a given tensor, finding its best rank-1 approximation, also known as finding the largest eigenvalue of a given tensor, has been studied in [30, 35, 10, 23, 21]. It should be noted that the CP-rank of a real-valued tensor can be different over $\mathbb{R}$ and $\mathbb{C}$; i.e., it may hold that $\mathrm{rank}_{CP}(\mathcal{F}) < \mathrm{rank}^{\mathbb{R}}_{CP}(\mathcal{F})$ for a real-valued tensor $\mathcal{F}$. For instance, a real-valued $2 \times 2 \times 2$ tensor $\mathcal{F}$ is given in [26] and it can be shown that $\mathrm{rank}^{\mathbb{R}}_{CP}(\mathcal{F}) = 3$ while $\mathrm{rank}_{CP}(\mathcal{F}) = 2$. In this paper, we shall focus on the notion of CP-rank in the complex domain and discuss two low-CP-rank tensor recovery problems: tensor completion and robust tensor recovery.

Low-CP-rank tensor recovery problem seeks to recover a low-CP-rank tensor based on limited observations. This problem can be formulated as

$$\min_{\mathcal{X} \in \mathbb{C}^{n_1 \times n_2 \cdots \times n_d}} \ \mathrm{rank}_{CP}(\mathcal{X}), \quad \text{s.t. } \boldsymbol{L}(\mathcal{X}) = b, \tag{2}$$

where $\boldsymbol{L} : \mathbb{C}^{n_1 \times n_2 \cdots \times n_d} \to \mathbb{C}^p$ is a linear mapping and $b \in \mathbb{C}^p$ denotes the observation to $\mathcal{X}$ under $\boldsymbol{L}$. Low-CP-rank tensor completion is a special case of (2) where the linear mapping in the constraint picks certain entries of the tensor. The low-CP-rank tensor completion can be formulated as

$$\min_{\mathcal{X} \in \mathbb{C}^{n_1 \times n_2 \cdots \times n_d}} \ \operatorname{rank}_{CP}(\mathcal{X}), \quad \text{s.t. } P_\Omega(\mathcal{X}) = P_\Omega(\mathcal{X}_0), \tag{3}$$

where $\mathcal{X}_0 \in \mathbb{C}^{n_1 \times n_2 \cdots \times n_d}$ is a given tensor, Ω is an index set and

$$[P_\Omega(\mathcal{X})]_{i_1, i_2, \ldots, i_d} = \begin{cases} \mathcal{X}_{i_1, i_2, \ldots, i_d}, & \text{if } (i_1, i_2, \ldots, i_d) \in \Omega, \\ 0, & \text{otherwise.} \end{cases}$$

In practice, the underlying low-CP-rank tensor data $\mathcal{F} \in \mathbb{C}^{n_1 \times n_2 \cdots \times n_d}$ may be heavily corrupted by a sparse noise tensor $\mathcal{Y}$. To identify and remove the noise, one can solve the following robust tensor recovery problem:

$$\min_{\mathcal{Y}, \mathcal{Z} \in \mathbb{C}^{n_1 \times n_2 \cdots \times n_d}} \ \operatorname{rank}_{CP}(\mathcal{Y}) + \lambda \|\mathcal{Z}\|_0, \quad \text{s.t. } \mathcal{Y} + \mathcal{Z} = \mathcal{F}, \tag{4}$$

where $\lambda > 0$ is a weighting parameter and $\|\mathcal{Z}\|_0$ denotes the number of nonzero entries of $\mathcal{Z}$.

Solving (2) and (4), however, are nearly impossible in practice. In fact, determining the CP-rank of a given tensor is known to be NP-hard in general [16]. Worse than solving a common NP-hard problem, computing the CP-rank for small size instances remains a difficult task. For example, a particular $9 \times 9 \times 9$ tensor is cited in [27] and its CP-rank is only known to be in between 18 and 23 to this date. The above low-CP-rank tensor recovery problems (2) and (4) thus appear to be quite hopeless. One way out of this dismay is to approximate the CP-rank by some more reasonable objects. Since computing the rank of a matrix is easy, a classical way for tackling tensor optimization problem is to unfold the tensor into certain matrix and then resort to some well established solution methods for matrix-rank optimization. A typical matrix unfolding technique is the so-called mode-n matricization [22]. Specifically, for a given tensor $\mathcal{F} \in \mathbb{C}^{n_1 \times n_2 \times \cdots \times n_d}$, we denote its mode-$n$ matricization by $F(n)$, which is obtained by arranging the n-th index of $\mathcal{F}$ as the column index of $F(n)$ and merging other indices of $\mathcal{F}$ as the row index of $F(n)$. The Tucker rank (or the mode-n-rank) of $\mathcal{F}$ is defined as the vector $(\operatorname{rank}(F(1)), \ldots, \operatorname{rank}(F(d)))$. For simplicity the averaged Tucker rank is often used and denoted as

$$\operatorname{rank}_n(\mathcal{F}) := \frac{1}{d} \sum_{j=1}^{d} \operatorname{rank}(F(j)).$$

As such, $\operatorname{rank}_n(\mathcal{F})$ is much easier to compute than the CP-rank, since it is just the average of d matrix ranks. Therefore, the following low-n-rank minimization model has been proposed for tensor recovery [31, 12]:

$$\min_{\mathcal{X} \in \mathbb{C}^{n_1 \times n_2 \cdots \times n_d}} \ \operatorname{rank}_n(\mathcal{X}), \quad \text{s.t. } \boldsymbol{L}(\mathcal{X}) = b, \tag{5}$$

where $\boldsymbol{L}$ and b are the same as the ones in (2). Since minimizing the rank function is still difficult, it was suggested in [31] and [12] to convexify the matrix rank function by the nuclear norm, which has become a common practice due to the seminal works on matrix rank minimization (see, e.g., [11, 37, 6, 7]). That is, the following convex optimization problem is solved instead of (5):

$$\min_{\mathcal{X}\in\mathbb{C}^{n_1\times n_2\cdots\times n_d}} \ \frac{1}{d}\sum_{j=1}^{d}\|X(j)\|_*, \quad \text{s.t. } \boldsymbol{L}(\mathcal{X}) = b, \tag{6}$$

where the nuclear norm $\|X(j)\|_*$ is defined as the sum of singular values of matrix $X(j)$. Some efficient algorithms such as alternating direction method of multipliers and Douglas-Rachford operator splitting methods were proposed in [31] and [12] to solve (6).

However, to the best of our knowledge, the relationship between the CP-rank and the Tucker rank of a tensor is still unclear so far, although it is easy to see (from similar argument as in Theorem 3.3) that the averaged Tucker rank is a lower bound for the CP-rank. In fact, there is a substantial gap between the averaged Tucker rank and the CP-rank. In Proposition 3.4, we present an $n\times n\times n\times n$ tensor whose CP-rank is n times the averaged Tucker rank. Moreover, in the numerical experiments we found two types of tensors whose CP-rank is strictly larger than the averaged Tucker rank; see Table 1 and Table 2. The theoretical guarantee of model (6) has been established in [41], which states that if the number of observations is at the order of $O(rn^{d-1})$, then with high probability the original tensor with Tucker rank $(r, r, \cdots, r)$ can be successfully recovered by solving (6). Therefore, model (6) and its variants have become popular in the area of tensor completion. However, our numerical results show that unless the CP-rank is extremely small, (6) usually fails to recover the tensor; see Table 3. As a different tensor unfolding technique, the square unfolding was proposed by Jiang et al. [21] for the tensor rank-one approximation problem. This technique was also considered by Mu et al. [34] for tensor completion problem. Mu et al. [34] showed that when the square unfolding is applied, the number of observations required to recover the tensor is of the order of $O(r^{\lfloor\frac{d}{2}\rfloor}n^{\lceil\frac{d}{2}\rceil})$, which is significantly less than the number needed by (6).

Convexifying the robust tensor recovery problem (4) was also studied by Tomioka et al. [40] and Goldfarb and Qin [14]. Specifically, they used the averaged Tucker rank to replace the CP-rank of $\mathcal{Y}$ and $\|\mathcal{Z}\|_1$ to replace $\|\mathcal{Z}\|_0$ in the objective of (4). However, we observe that this model cannot guarantee that the resulting solution $\mathcal{X}$ is of low CP-rank as shown in Table 6. Other works on this topic include [38, 24, 25]. Specifically, [38] compared the performance of the convex relaxation of the low-n-rank minimization model and the low-rank matrix completion model on applications in spectral image reconstruction. [24] proposed a Riemannian manifold optimization algorithm for finding a local optimum of the Tucker rank constrained optimization problem. [25] studied some adaptive sampling algorithms for low-rank tensor completion.

Note that in the definition of CP-rank, $\mathcal{F}\in\mathbb{C}^{n_1\times n_2\times\cdots\times n_d}$ is decomposed into the sum of asymmetric rank-one tensors. If $\mathcal{F}$ is a super-symmetric tensor, i.e., its component is invariant under any

permutation of the indices, then a natural extension is to decompose $\mathcal{F}$ into the sum of symmetric rank-one tensors, and this leads to the definition of symmetric CP-rank (see, e.g., [9]).

Definition 1.2 *Given a d-th order n-dimensional super-symmetric complex-valued tensor $\mathcal{F}$, its symmetric CP-rank (denoted by $\mathrm{rank}_{SCP}(\mathcal{F})$) is the smallest integer r such that*

$$\mathcal{F} = \sum_{i=1}^{r} \underbrace{a^i \otimes \cdots \otimes a^i}_{d},$$

with $a^i \in \mathbb{C}^n, i = 1, \ldots, r$.

It is obvious that $\mathrm{rank}_{CP}(\mathcal{F}) \leq \mathrm{rank}_{SCP}(\mathcal{F})$ for any given super-symmetric tensor $\mathcal{F}$. In the matrix case, i.e., when $d = 2$, it is known that the rank and symmetric rank are identical. However, in the higher order case, whether the CP-rank equals the symmetric CP-rank is still unknown, and this has become an interesting and challenging open problem (see Comon et al. [9]). There has been some recent progress towards answering this question. In particular, Zhang et al. [44] recently proved that $\mathrm{rank}_{CP}(\mathcal{F}) = \mathrm{rank}_{SCP}(\mathcal{F})$ holds for any d-th order super-symmetric tensor $\mathcal{F}$ if $\mathrm{rank}_{CP}(\mathcal{F}) \leq d$.

At this point, it is important to remark that the CP-rank stems from the idea of decomposing a general tensor into a sum of simpler – viz. *rank-one* in this context – tensors. The nature of the "simpler components" in the sum, however, can be made flexible and inclusive. In fact, in many cases it does not have to be a rank-one tensor as in the CP-decomposition. In particular, for a $2d$-th order tensor, the "simple tensor" being decomposed into could be the outer product of two tensors with lower degree, which is d in this paper, and we call this new decomposition the *M-decomposition*. It is easy to see (will be discussed in more details later) that after square unfolding each term in the M-decomposition is actually a rank-one matrix. Consequently, the minimum number of such simple terms can be regarded as a *rank*, or indeed the *M-rank* in our context, to be differentiated from other existing notions of tensor ranks. By imposing further symmetry on the "simple tensor" that composes the M-decomposition, the notion of symmetric M-decomposition (symmetric M-rank), and strongly symmetric M-decomposition (strongly symmetric M-rank) naturally follow. We will introduce the formal definitions later. The merits of the M-rank are twofold. First, for some structured tensors, we can show, through either theoretical analysis or numerical verifications, that the M-rank is much better than the averaged Tucker rank in terms of approximating the CP-rank. Second, for low-CP-rank tensor recovery problems, the low-M-rank approach can improve the recoverability and our numerical tests suggest that the M-rank remain a good approximation to the CP-rank even in the presence of some gross errors.

The main contributions of this paper are as follows. First, we introduce several new notions of tensor decomposition for the even order tensors, followed by the new notions of tensor M-rank,

symmetric M-rank and strongly symmetric M-rank. Second, we prove the equivalence of these three rank definitions for even-order super-symmetric tensors. Third, we establish the connection between these new ranks and the CP-rank and symmetric CP-rank. Specifically, we show that for a fourth-order tensor, both the CP-rank and the symmetric CP-rank can be lower and upper bounded (up to a constant factor) by the M-rank, and the bound is tight for asymmetric tensors. As a byproduct, we present a class of tensors whose CP-rank can be exactly computed easily. Finally, in the numerical experiments we show that the M-rank is equal to the CP-rank in all randomly generated instances and the low-M-rank method outperforms the low-n-rank approach for both tensor completion and robust tensor recovery problems.

Notation. We use $\mathbb{C}^n$ to denote the n-dimensional complex-valued vector space. We adopt calligraphic letter to denote a tensor, i.e. $\mathcal{A} = (\mathcal{A}_{i_1 i_2 \cdots i_d})_{n_1 \times n_2 \times \cdots \times n_d}$. $\mathbb{C}^{n_1 \times n_2 \times \cdots \times n_d}$ denotes the space of d-th order $n_1 \times n_2 \times \cdots \times n_d$ dimensional complex-valued tensor. $\pi(i_1, i_2, \cdots, i_d)$ denotes a permutation of indices $(i_1, i_2, \cdots, i_d)$. We use $\mathcal{A}_\pi$ to denote the tensor obtained by permuting the indices of $\mathcal{A}$ according to permutation π. Formally speaking, a tensor $\mathcal{F} \in \mathbb{C}^{n_1 \times n_2 \times \cdots \times n_d}$ is called super-symmetric, if $n_1 = n_2 = \ldots = n_d$ and $\mathcal{F} = \mathcal{F}_\pi$ for any permutation π. The space where $\underbrace{n \times n \times \cdots \times n}_{d}$ super-symmetric tensors reside is denoted by $\mathbb{S}^{n^d}$. We use $\otimes$ to denote the outer product of two tensors; that is, for $\mathcal{A}_1 \in \mathbb{C}^{n_1 \times n_2 \times \cdots \times n_d}$ and $\mathcal{A}_2 \in \mathbf{R}^{n_{d+1} \times n_{d+2} \times \cdots \times n_{d+\ell}}$, $\mathcal{A}_1 \otimes \mathcal{A}_2 \in \mathbb{C}^{n_1 \times n_2 \times \cdots \times n_{d+\ell}}$ and

$$(\mathcal{A}_1 \otimes \mathcal{A}_2)_{i_1 i_2 \cdots i_{d+\ell}} = (\mathcal{A}_1)_{i_1 i_2 \cdots i_d} (\mathcal{A}_2)_{i_{d+1} \cdots i_{d+\ell}}.$$

2 M-rank, symmetric M-rank and strongly symmetric M-rank

In this section, we shall introduce the M-decomposition (correspondingly M-rank), the symmetric M-decomposition (correspondingly symmetric M-rank), and the strongly symmetric M-decomposition (correspondingly strongly symmetric M-rank) for tensors, which will be used to provide lower and upper bounds for the CP-rank and the symmetric CP-rank.

2.1 The M-rank of even-order tensor

The M-decomposition of an even-order tensor is defined as follows.

Definition 2.1 *For an even-order tensor $\mathcal{F} \in \mathbb{C}^{n_1 \times n_2 \cdots \times n_{2d}}$, the M-decomposition is to find some tensors $\mathcal{A}^i \in \mathbb{C}^{n_1 \times \cdots \times n_d}$, $\mathcal{B}^i \in \mathbb{C}^{n_{d+1} \times \cdots \times n_{2d}}$ with $i = 1, \ldots, r$ such that*

$$\mathcal{F} = \sum_{i=1}^{r} \mathcal{A}^i \otimes \mathcal{B}^i. \tag{7}$$

The motivation for studying this decomposition stems from the following novel matricization technique called square unfolding that has been considered in [21, 34, 43].

Definition 2.2 *The square unfolding of an even-order tensor* $\mathcal{F} \in \mathbb{C}^{n_1 \times n_2 \cdots \times n_{2d}}$ *(denoted by* $\boldsymbol{M}(\mathcal{F}) \in \mathbb{C}^{(n_1 \cdots n_d) \times (n_{d+1} \cdots n_{2d})}$*) is a matrix that is defined as*

$$\boldsymbol{M}(\mathcal{F})_{k\ell} := \mathcal{F}_{i_1 \cdots i_{2d}},$$

where

$$k = \sum_{j=2}^{d} (i_j - 1) \prod_{q=1}^{j-1} n_q + i_1,\ 1 \le i_j \le n_j,\ 1 \le j \le d,$$

$$\ell = \sum_{j=d+2}^{2d} (i_j - 1) \prod_{q=d+1}^{j-1} n_q + i_{d+1},\ 1 \le i_j \le n_j,\ d+1 \le j \le 2d.$$

In Definition 2.2, the square unfolding merges d indices of $\mathcal{F}$ as the row index of $\boldsymbol{M}(\mathcal{F})$, and merges the other d indices of $\mathcal{F}$ as the column index of $\boldsymbol{M}(\mathcal{F})$. In this pattern of unfolding, we can see that the M-decomposition (7) can be rewritten as

$$\boldsymbol{M}(\mathcal{F}) = \sum_{i=1}^{r} \mathbf{a}^i (\mathbf{b}^i)^\top,$$

where $a^i = \boldsymbol{V}(\mathcal{A}^i)$, $b^i = \boldsymbol{V}(\mathcal{B}^i)$ for $i = 1, \ldots, r$, and $\boldsymbol{V}(\cdot)$ is the vectorization operator. Specifically, for a given tensor $\mathcal{F} \in C^{n_1 \times n_2 \cdots \times n_d}$, $\boldsymbol{V}(\mathcal{F})_k := \mathcal{F}_{i_1 \cdots i_d}$ with

$$k = \sum_{j=2}^{d} (i_j - 1) \prod_{q=1}^{j-1} n_q + i_1, 1 \le i_j \le n_j, 1 \le j \le d.$$

Therefore, the M-decomposition of $\mathcal{F}$ is exactly the rank-one decomposition of the matrix $\boldsymbol{M}(\mathcal{F})$. Apparently, unless $\mathcal{F}$ is super-symmetric, there are different ways to unfold the tensor $\mathcal{F}$ by permuting the $2d$ indices. Taking this into account, we now define two types of M-rank (namely, M$^+$-rank and M$^-$-rank) of an even-order tensor as follows.

Definition 2.3 *Given an even-order tensor* $\mathcal{F} \in \mathbb{C}^{n_1 \times n_2 \cdots \times n_{2d}}$*, its* M^-*-rank (denoted by* $\operatorname{rank}_{M^-}(\mathcal{F})$*) is the smallest rank of all possible square unfolding matrices, i.e.,*

$$\operatorname{rank}_{M^-}(\mathcal{F}) = \min_{\pi \in \Pi(1,\ldots,2d)} \operatorname{rank}\left(\boldsymbol{M}(\mathcal{F}_\pi)\right), \tag{8}$$

where $\Pi(1, \ldots, 2d)$ *denotes the set of all possible permutations of indices* $(1, \ldots, 2d)$*, and* $\mathcal{F}_\pi$ *is the tensor obtained by permuting the indices of* $\mathcal{F}$ *according to permutation* π*. In other words,* $\operatorname{rank}_{M^-}(\mathcal{F})$ *is the smallest integer* r *such that*

$$\mathcal{F}_\pi = \sum_{i=1}^{r} \mathcal{A}^i \otimes \mathcal{B}^i, \tag{9}$$

holds for some permutation $\pi \in \Pi(1,\ldots,2d)$, $\mathcal{A}^i \in \mathbb{C}^{n_{i_1}\times\cdots\times n_{i_d}}$, $\mathcal{B}^i \in \mathbb{C}^{n_{i_{d+1}}\times\cdots\times n_{i_{2d}}}$ *with* $(i_1,\cdots,i_{2d}) = \pi(1,\cdots,2d)$, $i = 1,\ldots,r$. *Similarly, the* M^+*-rank (denoted by* $\mathrm{rank}_{M^+}(\mathcal{F})$*) is defined as the largest rank of all possible square unfolding matrices:*

$$\mathrm{rank}_{M^+}(\mathcal{F}) = \max_{\pi\in\Pi(1,\ldots,2d)} \mathrm{rank}\left(\boldsymbol{M}(\mathcal{F}_\pi)\right). \tag{10}$$

2.2 Symmetric M-rank and strongly symmetric M-rank of even-order super-symmetric tensor

Note that if $\mathcal{F}$ is an even-order super-symmetric tensor, $\mathrm{rank}_{M^+}(\mathcal{F}) = \mathrm{rank}_{M^-}(\mathcal{F})$. In this case, we can simplify the notation without causing any confusion by using $\mathrm{rank}_M(\mathcal{F})$ to denote the M-rank of $\mathcal{F}$.

As we mentioned earlier, the decomposition (7) is essentially based on the matrix rank-one decomposition of the matricized tensor. In the matrix case, it is clear that there are different ways to decompose a symmetric matrix; for instance,

$$ab^\top + ba^\top = \frac{1}{2}(a+b)(a+b)^\top - \frac{1}{2}(a-b)(a-b)^\top.$$

In other words, a given symmetric matrix may be decomposed as a sum of *symmetric* rank-one terms, as well as a sum of *non-symmetric* rank-one terms, however yielding the same rank: the minimum number of respective decomposed terms. A natural question arises when dealing with tensors: Does the same property hold for the super-symmetric tensors? The M-decomposition of a tensor is in fact subtler: the decomposed terms can be restricted to symmetric products, and they can also be further restricted to be super-symmetric.

Therefore, we can define the symmetric M-rank and the strongly symmetric M-rank for even-order super-symmetric tensor as follows.

Definition 2.4 *For an even-order super-symmetric tensor* $\mathcal{F} \in \mathbb{S}^{n^{2d}}$*, its symmetric M-decomposition is defined as*

$$\mathcal{F} = \sum_{i=1}^r \mathcal{B}^i \otimes \mathcal{B}^i, \quad \mathcal{B}^i \in \mathbb{C}^{n^d}, i = 1,\ldots,r. \tag{11}$$

The symmetric M-rank of $\mathcal{F}$ *(denoted by* $\mathrm{rank}_{SM}(\mathcal{F})$*) is the rank of the symmetric matrix* $\boldsymbol{M}(\mathcal{F})$*, i.e.,* $\mathrm{rank}_{SM}(\mathcal{F}) = \mathrm{rank}\left(\boldsymbol{M}(\mathcal{F})\right) = \mathrm{rank}_M(\mathcal{F})$*; or equivalently* $\mathrm{rank}_{SM}(\mathcal{F})$ *is the smallest integer* r *such that* (11) *holds.*

In a similar vein, the strongly symmetric M-decomposition is defined as

$$\mathcal{F} = \sum_{i=1}^r \mathcal{A}^i \otimes \mathcal{A}^i, \quad \mathcal{A}^i \in \mathbb{S}^{n^d}, i = 1,\ldots,r, \tag{12}$$

and the strongly symmetric M-rank of $\mathcal{F}$ (denoted by $\operatorname{rank}_{SSM}(\mathcal{F})$) is defined as the smallest integer r such that (12) *holds.*

The fact that the M-rank and the symmetric M-rank of an even-order super-symmetric tensor are always equal follows from the similar property of the symmetric matrices. (Note however the M-decompositions may be different). Interestingly, we can show that $\operatorname{rank}_{SM}(\mathcal{F}) = \operatorname{rank}_{SSM}(\mathcal{F})$ for any even-order super-symmetric tensor $\mathcal{F}$, which appears to be a new property of the super-symmetric even-order tensors.

2.3 Equivalence of symmetric M-rank and strongly symmetric M-rank

To show the equivalence of the symmetric M-rank and the strongly symmetric M-rank, we need to introduce the concept of partial symmetric tensors and some lemmas first.

Definition 2.5 *We say a tensor $\mathcal{F} \in \mathbb{C}^{n^d}$ partial symmetric with respect to indices $\{1, \ldots, m\}$, $m < d$, if*

$$\mathcal{F}_{i_1,\ldots,i_m,i_{m+1},\ldots,i_d} = \mathcal{F}_{\pi(i_1,\ldots,i_m),i_{m+1},\ldots,i_d}, \quad \forall \pi \in \Pi(1, \ldots, m).$$

We use $\pi_{i,j} \in \Pi(1, \cdots, d)$ to denote the specific permutation that exchanges the i-th and the j-th indices and keeps other indices unchanged.

The following result holds directly from Definition 2.5.

Lemma 2.1 *Suppose tensor $\mathcal{F} \in \mathbb{C}^{n^d}$ is partial symmetric with respect to indices $\{1, \ldots, m\}$, $m < d$. Then the tensor*

$$\mathcal{F} + \sum_{j=1}^{m} \mathcal{F}_{\pi_{j,m+1}}$$

is partial symmetric with respect to indices $\{1, \ldots, m+1\}$. Moreover, it is easy to verify that for $\ell \le k \le m$,

$$\begin{aligned} \left(\sum_{j=1}^{k} \left(\mathcal{F} - \mathcal{F}_{\pi_{j,m+1}} \right) \right)_{\pi_{\ell,m+1}} &= k \cdot \mathcal{F}_{\pi_{\ell,m+1}} - \sum_{j \neq \ell} \mathcal{F}_{\pi_{j,m+1}} - \mathcal{F} \\ &= -k \left(\mathcal{F} - \mathcal{F}_{\pi_{\ell,m+1}} \right) + \sum_{j \neq \ell} \left(\mathcal{F} - \mathcal{F}_{\pi_{j,m+1}} \right). \end{aligned} \tag{13}$$

We are now ready to prove the following key lemma.

Lemma 2.2 *Suppose $\mathcal{F} \in \mathbb{S}^{n^{2d}}$ and*

$$\mathcal{F} = \sum_{i=1}^{r} \mathcal{B}^i \otimes \mathcal{B}^i, \text{ where } \mathcal{B}^i \in \mathbb{C}^{n^d} \text{ is partial symmetric with respect to } \{1, \ldots, m\}, m < d.$$

Then there exist tensors $\mathcal{A}^i \in \mathbb{C}^{n^d}$, which is partial symmetric with respect to $\{1, \ldots, m+1\}$, for $i = 1, \ldots, r$, such that

$$\mathcal{F} = \sum_{i=1}^{r} \mathcal{A}^i \otimes \mathcal{A}^i.$$

Proof. Define $\mathcal{A}^i = \frac{1}{m+1}\left(\mathcal{B}^i + \sum\limits_{j=1}^{m} \mathcal{B}^i_{\pi_{j,m+1}}\right)$. From Lemma 2.1 we know that $\mathcal{A}^i$ is partial symmetric with respect to $\{1, \ldots, m+1\}$, for $i = 1, \ldots, r$. It is easy to show that

$$\mathcal{B}^i = \mathcal{A}^i + \sum_{j=1}^{m} \mathcal{C}^i_j, \text{ with } \quad \mathcal{C}^i_j = \frac{1}{m+1}\left(\mathcal{B}^i - \mathcal{B}^i_{\pi_{j,m+1}}\right).$$

Because $\mathcal{F}$ is super-symmetric, we have $\mathcal{F} = \mathcal{F}_{\pi_{d+1,d+m+1}} = \mathcal{F}_{\pi_{1,m+1}} = (\mathcal{F}_{\pi_{1,m+1}})_{\pi_{d+1,d+m+1}}$, which implies

$$\mathcal{F} = \sum_{i=1}^{r} \mathcal{B}^i \otimes \mathcal{B}^i = \sum_{i=1}^{r} \mathcal{B}^i \otimes \mathcal{B}^i_{\pi_{1,m+1}} = \sum_{i=1}^{r} \mathcal{B}^i_{\pi_{1,m+1}} \otimes \mathcal{B}^i = \sum_{i=1}^{r} \mathcal{B}^i_{\pi_{1,m+1}} \otimes \mathcal{B}^i_{\pi_{1,m+1}}. \tag{14}$$

By using (13), we have

$$\mathcal{B}^i_{\pi_{1,m+1}} = \left(\mathcal{A}^i + \sum_{j=1}^{m} \mathcal{C}^i_j\right)_{\pi_{1,m+1}} = \mathcal{A}^i + \sum_{j=2}^{m} \mathcal{C}^i_j - m \cdot \mathcal{C}^i_1. \tag{15}$$

Combining (15) and (14) yields

$$\begin{aligned}
\mathcal{F} &= \sum_{i=1}^{r} \mathcal{B}^i \otimes \mathcal{B}^i = \sum_{i=1}^{r} \left(\mathcal{A}^i + \sum_{j=1}^{m} \mathcal{C}^i_j\right) \otimes \left(\mathcal{A}^i + \sum_{j=1}^{m} \mathcal{C}^i_j\right) && (16)\\
&= \sum_{i=1}^{r} \mathcal{B}^i \otimes \mathcal{B}^i_{\pi_{1,m+1}} = \sum_{i=1}^{r} \left(\mathcal{A}^i + \sum_{j=1}^{m} \mathcal{C}^i_j\right) \otimes \left(\mathcal{A}^i + \sum_{j=2}^{m} \mathcal{C}^i_j - m \cdot \mathcal{C}^i_1\right) && (17)\\
&= \sum_{i=1}^{r} \mathcal{B}^i_{\pi_{1,m+1}} \otimes \mathcal{B}^i = \sum_{i=1}^{r} \left(\mathcal{A}^i + \sum_{j=2}^{m} \mathcal{C}^i_j - m \cdot \mathcal{C}^i_1\right) \otimes \left(\mathcal{A}^i + \sum_{j=1}^{m} \mathcal{C}^i_j\right) && (18)\\
&= \sum_{i=1}^{r} \mathcal{B}^i_{\pi_{1,m+1}} \otimes \mathcal{B}^i_{\pi_{1,m+1}} = \sum_{i=1}^{r} \left(\mathcal{A}^i + \sum_{j=2}^{m} \mathcal{C}^i_j - m \cdot \mathcal{C}^i_1\right) \otimes \left(\mathcal{A}^i + \sum_{j=2}^{m} \mathcal{C}^i_j - m \cdot \mathcal{C}^i_1\right). && (19)
\end{aligned}$$

It is easy to check that

$$\frac{(19) + m \times (18) + m \times (17) + m^2 \times (16)}{(1+m)^2} \implies \mathcal{F} = \sum_{i=1}^{r} \left(\mathcal{A}^i + \sum_{j=2}^{m} \mathcal{C}_j^i \right) \otimes \left(\mathcal{A}^i + \sum_{j=2}^{m} \mathcal{C}_j^i \right).$$

Then we repeat this procedure. That is, since $\mathcal{F} \in \mathbb{S}^{n^{2d}}$, we have $\mathcal{F} = \mathcal{F}_{\pi_{d+2,d+m+1}} = \mathcal{F}_{\pi_{2,m+1}} = (\mathcal{F}_{\pi_{2,m+1}})_{\pi_{d+2,d+m+1}}$. By letting $\mathcal{B}^i = \mathcal{A}^i + \sum\limits_{j=2}^{d} \mathcal{C}_j^i$, we can apply the same procedure as above to obtain $\mathcal{F} = \sum_{i=1}^{r} \left(\mathcal{A}^i + \sum\limits_{j=3}^{m} \mathcal{C}_j^i \right) \otimes \left(\mathcal{A}^i + \sum\limits_{j=3}^{m} \mathcal{C}_j^i \right)$. We just repeat this procedure until $\mathcal{F} = \sum_{i=1}^{r} \mathcal{A}^i \otimes \mathcal{A}^i$ and this completes the proof. □

Now we are ready to present the equivalence of symmetric M-rank and strongly symmetric M-rank.

Theorem 2.3 *For an even-order super-symmetric tensor $\mathcal{F} \in \mathbb{S}^{n^{2d}}$, its M-rank, symmetric M-rank and strongly symmetric M-rank are the same, i.e.* $\text{rank}_M(\mathcal{F}) = \text{rank}_{SM}(\mathcal{F}) = \text{rank}_{SSM}(\mathcal{F})$.

Proof. The equality $\text{rank}_M(\mathcal{F}) = \text{rank}_{SM}(\mathcal{F})$ follows directly from the definition of symmetric M-rank. We thus only need to prove $\text{rank}_{SM}(\mathcal{F}) = \text{rank}_{SSM}(\mathcal{F})$. Suppose $\text{rank}_{SM}(\mathcal{F}) = r$, which means there exist $\mathcal{B}^i \in \mathbb{C}^{n^d}, i = 1, \ldots, r$, such that $\mathcal{F} = \sum\limits_{i=1}^{r} \mathcal{B}^i \otimes \mathcal{B}^i$. By applying Lemma 2.2 at most d times, we can find super-symmetric tensors $\mathcal{A}^i \in \mathbb{S}^{n^d}, i = 1, \ldots, r$ such that $\mathcal{F} = \sum\limits_{i=1}^{r} \mathcal{A}^i \otimes \mathcal{A}^i$. Hence, we have $\text{rank}_{SSM}(\mathcal{F}) \le r = \text{rank}_{SM}(\mathcal{F})$. On the other hand, it is obvious that $\text{rank}_{SM}(\mathcal{F}) \le \text{rank}_{SSM}(\mathcal{F})$. Combining these two inequalities yields $\text{rank}_{SM}(\mathcal{F}) = \text{rank}_{SSM}(\mathcal{F})$. □

3 Bounding CP-rank for even-order tensor using M-rank

In this section, we analyze the relation between the CP-rank and the M-rank. Specifically, for even-order tensor, we establish the equivalence between the symmetric CP-rank and the M-rank under the rank-one assumption. Then we particularly focus on the fourth-order tensors, because many multi-dimensional data from real practice are in fact fourth-order tensors. For example, the colored video completion and decomposition problems considered in [12, 14, 31] can be formulated as low-CP-rank fourth-order tensor recovery problems. We show that the CP-rank and the symmetric CP-rank for fourth-order tensor can be both lower and upper bounded (up to a constant factor) by the corresponding M-rank.

3.1 Rank-one equivalence for super-symmetric even-order tensor

In our previous work [21], we already showed that if a super-symmetric even-order tensor $\mathcal{F}$ is real-valued and the decomposition is performed in the real domain, then $\mathrm{rank}_{CP}(\mathcal{F}) = 1 \iff \mathrm{rank}(\boldsymbol{M}(\mathcal{F})) = 1$. Here we show that a similar result can be established when $\mathcal{F}$ is complex-valued and the decomposition is performed in the complex domain. To see this, we first present the following lemma.

Lemma 3.1 *If a d-th order tensor $\mathcal{A} = a^1 \otimes a^2 \otimes \cdots \otimes a^d \in \mathbb{S}^{n^d}$ is super-symmetric, then we have $a^i = \pm a^1$ for $i = 2, \ldots, d$ and $\mathcal{A} = \underbrace{b \otimes b \otimes \cdots \otimes b}_{d}$ for some $b \in \mathbb{C}^n$.*

Proof. Since $\mathcal{A}$ is super-symmetric, construct $\mathcal{T} = \bar{\mathcal{A}} \otimes \mathcal{A}$ and it is easy to show that

$$\mathcal{T}_{i_1 \ldots i_d i_{d+1} \ldots i_{2d}} = \mathcal{T}_{j_1 \ldots j_d j_{d+1} \ldots j_{2d}}, \ \forall \, (j_1 \ldots j_d) \in \Pi(i_1 \ldots i_d), \, (j_{d+1} \ldots j_{2d}) \in \Pi(i_{d+1} \ldots i_{2d}),$$

and

$$\mathcal{T}_{i_1 \ldots i_d i_{d+1} \ldots i_{2d}} = \overline{\mathcal{T}_{i_{d+1} \ldots i_{2d} i_1 \ldots i_d}}, \ \forall 1 \le i_1 \le \cdots \le i_d \le n, \, 1 \le i_{d+1} \le \cdots \le i_{2d} \le n.$$

Therefore, $\mathcal{T}$ belongs to the so-called conjugate partial symmetric tensor introduced in [20]. Moreover, from Theorem 6.5 in [20], we know that

$$\max_{\|x\|=1} \mathcal{T}(\underbrace{\overline{x}, \ldots, \overline{x}}_{d}, \underbrace{x, \ldots, x}_{d}) = \max_{\|x^i\|=1, \, i=1,\ldots,d} \mathcal{T}(\overline{x^1}, \ldots, \overline{x^d}, x^1, \ldots, x^d) = \|a^1\|^2 \cdot \|a^2\|^2 \cdot \cdots \cdot \|a^d\|^2.$$

So there must exist an $\hat{x}$ with $\|\hat{x}\| = 1$ such that $|(a^i)^\top \hat{x}| = \|a^i\|$ for all i, which implies that $a^i = \pm a^1$ for $i = 2, \ldots, d$, and $\mathcal{A} = \lambda \underbrace{a^1 \otimes a^1 \otimes \cdots \otimes a^1}_{d}$ for some $\lambda = \pm 1$. Finally by taking $b = \sqrt[d]{\lambda} a^1$, the conclusion follows. $\square$

The rank-one equivalence is established in the following theorem.

Theorem 3.2 *Suppose $\mathcal{F} \in \mathbb{S}^{n^{2d}}$ and we have*

$$\mathrm{rank}_M(\mathcal{F}) = 1 \iff \mathrm{rank}_{SCP}(\mathcal{F}) = 1.$$

Proof. Suppose $\mathrm{rank}_{SCP}(\mathcal{F}) = 1$ and $\mathcal{F} = \underbrace{x \otimes \cdots \otimes x}_{2d}$ for some $x \in \mathbb{C}^n$. By constructing $\mathcal{A} = \underbrace{x \otimes \cdots \otimes x}_{d}$, we have $\mathcal{F} = \mathcal{A} \otimes \mathcal{A}$ with $\mathcal{A} \in \mathbb{S}^{n^d}$. Thus, $\mathrm{rank}_M(\mathcal{F}) = 1$.

To prove the other direction, suppose that we have $\mathcal{F} \in \mathbb{S}^{n^{2d}}$ and its M-rank is one, i.e. $\mathcal{F} = \mathcal{A} \otimes \mathcal{A}$ for some $\mathcal{A} \in \mathbb{S}^{n^d}$. By similar arguments as in Lemma 2.1 and Proposition 2.3 in [21], one has

that the Tucker rank of $\mathcal{A}$ is $(1, 1, \cdots, 1)$ and consequently the asymmetric CP-rank of $\mathcal{A}$ is one. This fact together with Lemma 3.1 implies that the symmetric CP-rank of $\mathcal{A}$ is one as well, i.e., $\mathcal{A} = \underbrace{b \otimes \cdots \otimes b}_{d}$ for some $b \in \mathbb{C}^n$. It follows from $\mathcal{F} = \mathcal{A} \otimes \mathcal{A} = \underbrace{b \otimes \cdots \otimes b}_{2d}$ that $\mathrm{rank}_{SCP}(\mathcal{F}) = 1$. □

3.2 Bound for asymmetric fourth-order tensor

For an asymmetric fourth-order tensor, the relation between its CP-rank and the corresponding M-rank is summarized in the following result.

Theorem 3.3 *Suppose $\mathcal{F} \in \mathbb{C}^{n_1 \times n_2 \times n_3 \times n_4}$ with $n_1 \le n_2 \le n_3 \le n_4$. Then for any permutation π of $(1, 2, 3, 4)$ it holds that*

$$\mathrm{rank}(\boldsymbol{M}(\mathcal{F}_\pi)) \le \mathrm{rank}_{CP}(\mathcal{F}_\pi) \le n_1 n_3 \cdot \mathrm{rank}(\boldsymbol{M}(\mathcal{F}_\pi)). \tag{20}$$

Moreover, the inequalities above can be sharpened to

$$\mathrm{rank}_{M^+}(\mathcal{F}) \le \mathrm{rank}_{CP}(\mathcal{F}) \le n_1 n_3 \cdot \mathrm{rank}_{M^-}(\mathcal{F}). \tag{21}$$

Proof. Suppose $\mathrm{rank}_{CP}(\mathcal{F}) = r$. Let the rank-one decomposition be

$$\mathcal{F} = \sum_{i=1}^{r} a^{1,i} \otimes a^{2,i} \otimes a^{3,i} \otimes a^{4,i} \ \text{ with } \ a^{k,i} \in \mathbb{C}^{n_i} \text{ for } k = 1, \ldots, 4 \text{ and } i = 1, \ldots, r.$$

By letting $A^i = a^{1,i} \otimes a^{2,i}$ and $B^i = a^{3,i} \otimes a^{4,i}$, we get $\mathcal{F} = \sum_{i=1}^r A^i \otimes B^i$. Thus $\mathrm{rank}_M(\mathcal{F}) \le r = \mathrm{rank}_{CP}(\mathcal{F})$. In fact, this holds for $\mathcal{F}_\pi$ where π is any permutation of $(1, 2, 3, 4)$.

On the other hand, for any $\mathcal{F}_\pi$ denote $r_M = \mathrm{rank}(\boldsymbol{M}(\mathcal{F}_\pi))$ and $(j_1, j_2, j_3, j_4) = \pi(1, 2, 3, 4)$. Then

$$\mathcal{F}_\pi = \sum_{i=1}^{r_M} A^i \otimes B^i \ \text{ with matrices } \ A^i \in \mathbb{C}^{n_{j_1} \times n_{j_2}}, B^i \in \mathbb{C}^{n_{j_3} \times n_{j_4}} \text{ for } i = 1, \ldots, r_M,$$

and it follows that $\mathrm{rank}(A^i) \le \ell_1$ and $\mathrm{rank}(B^i) \le \ell_2$ for all $i = 1, \ldots, r_M$, where $\ell_1 := \min\{n_{j_1}, n_{j_2}\}$ and $\ell_2 := \min\{n_{j_3}, n_{j_4}\}$. In other words, matrices A^i and B^i admit some rank-one decompositions with at most ℓ_1 and ℓ_2 terms, respectively. Consequently, $\mathcal{F}$ can be decomposed as the sum of at most $r_M \ell_1 \ell_2$ rank-one tensors, or equivalently

$$\mathrm{rank}_{CP}(\mathcal{F}_\pi) \le \min\{n_{j_1}, n_{j_2}\} \cdot \min\{n_{j_3}, n_{j_4}\} \cdot \mathrm{rank}_M(\mathcal{F}_\pi) \le n_1 n_3 \cdot \mathrm{rank}_M(\mathcal{F}_\pi).$$

Since the bounds (20) hold for all $\mathcal{F}_\pi$ and

$$\mathrm{rank}_{M^-}(\mathcal{F}) = \min_\pi \mathrm{rank}(\boldsymbol{M}(\mathcal{F}_\pi)), \ \mathrm{rank}_{M^+}(\mathcal{F}) = \max_\pi \mathrm{rank}(\boldsymbol{M}(\mathcal{F}_\pi)),$$

the sharper bounds (21) follow immediately. $\square$

The following results further show that the bounds in (21) are actually tight.

Proposition 3.4 *Let us consider a fourth order tensor $\mathcal{F} \in \mathbb{C}^{n_1 \times n_2 \times n_3 \times n_4}$ such that*

$$\mathcal{F} = A \otimes B \text{ for some matrices } A \in \mathbb{C}^{n_1 \times n_2} \text{ and } B \in \mathbb{C}^{n_3 \times n_4}. \tag{22}$$

Denote $r_1 = \operatorname{rank}(A)$, $r_2 = \operatorname{rank}(B)$. Then, the following holds:

(i) The Tucker rank of $\mathcal{F}$ is (r_1, r_1, r_2, r_2);

(ii) $\operatorname{rank}_{M^+}(\mathcal{F}) = r_1 r_2$ and $\operatorname{rank}_{M^-}(\mathcal{F}) = 1$;

(iii) $\operatorname{rank}_{CP}(\mathcal{F}) = r_1 r_2$.

Proof. Suppose the singular value decompositions of A and B are given by

$$A = \sum_{i=1}^{r_1} a^i \otimes b^i \text{ and } B = \sum_{j=1}^{r_2} c^j \otimes d^j. \tag{23}$$

Recall that $F(1)$ denotes the mode-1 unfolding of $\mathcal{F}$. According to (23), it is easy to verify that

$$F(1) = \sum_{i=1}^{r_1} a^i \otimes \boldsymbol{V}(b^i \otimes B). \tag{24}$$

Moreover we observe that for $i \neq j$, $(\boldsymbol{V}(b^i \otimes B))^\top (\boldsymbol{V}(b^j \otimes B)) = (b^i)^\top b^j \cdot \operatorname{tr}(B^\top B) = 0$. Thus, (24) is indeed an orthogonal decomposition of $F(1)$ and thus, $\operatorname{rank}(F(1)) = r_1$. Similarly we can show that $\operatorname{rank}(F(2)) = r_1$, $\operatorname{rank}(F(3)) = r_2$ and $\operatorname{rank}(F(4)) = r_2$. This proves part (i).

Now we consider the square unfoldings of $\mathcal{F}$. Let $F(1,2)$, $F(1,3)$, $F(1,4)$ be the square unfolded matrices by grouping indices $(1,2)$, $(1,3)$, $(1,4)$ respectively. Due to (22), we immediately have that $\operatorname{rank}(F(1,2)) = 1$ and also

$$F(1,3) = \sum_{i=1}^{r_1} \sum_{j=1}^{r_2} \boldsymbol{V}(a^i \otimes c^j) \otimes \boldsymbol{V}(b^i \otimes d^j). \tag{25}$$

From the orthogonality of a^i's, b^i's, c^j's, and d^j's, it follows that $\{\boldsymbol{V}(a^i \otimes c^j)\}_{i,j}$ and $\{\boldsymbol{V}(b^i \otimes d^j)\}_{i,j}$ are two orthogonal bases. In other words, (25) is an orthogonal decomposition of $F(1,3)$ and thus $\operatorname{rank}(F(1,3)) = r_1 r_2$. In the same vein we can show that $\operatorname{rank}(F(1,4)) = r_1 r_2$ as well. Since $F(1,2)$, $F(1,3)$ and $F(1,4)$ form all the square unfoldings of $\mathcal{F}$, we can conclude that $\operatorname{rank}_{M^+}(\mathcal{F}) = \operatorname{rank}(F(1,3)) = \operatorname{rank}(F(1,4)) = r_1 r_2$ and $\operatorname{rank}_{M^-}(\mathcal{F}) = 1$. This proves part (ii).

Finally, since (22) also implies

$$\mathcal{F} = \sum_{i=1}^{r_1}\sum_{j=1}^{r_2} a^i \otimes b^i \otimes c^j \otimes d^j,$$

it then follows that $\text{rank}_{CP}(\mathcal{F}) \le r_1\, r_2$. On the other hand, note that the first inequality in (20) holds for any square unfoldings. Combining this with $\text{rank}(F(1,3)) = r_1\, r_2$, one concludes that $r_1\, r_2$ is a lower bound of $\text{rank}_{CP}\mathcal{F}$ as well, hence $\text{rank}_{CP}(\mathcal{F}) = r_1\, r_2$. $\square$

Remark 3.5 *Now we remark that the bounds in* (21) *are tight. Suppose tensor $\mathcal{F}$ is given in the form of* (22) *with $n_1 \le n_2 \le n_3 \le n_4$, and $\text{rank}(A) = n_1$, $\text{rank}(B) = n_3$. According to the above results, we have $\text{rank}_{M^-}(\mathcal{F}) = 1$ and $\text{rank}_{M^+}(\mathcal{F}) = \text{rank}_{CP}(\mathcal{F}) = n_1 n_3$, which imply that the both lower bound and upper bound in* (21) *are essentially tight. Moreover, in this example the Tucker rank is exactly (n_1, n_1, n_3, n_3), which in turn shows that rank_{M^+} is a superior approximation of rank_{CP} at least for this example. In the numerical part, we shall present more examples for which rank_{CP} is strictly larger than any component of Tucker rank but is essentially identical to rank_{M^+}. In addition, it is easy to show that by similar argument the bounds in* (20) *also hold for the ranks defined for real-valued decompositions, i.e., $\text{rank}^{\mathbb{R}}(\boldsymbol{M}(\mathcal{F}_\pi)) \le \text{rank}^{\mathbb{R}}_{CP}(\mathcal{F}_\pi) \le n_1 n_3 \cdot \text{rank}^{\mathbb{R}}(\boldsymbol{M}(\mathcal{F}_\pi))$. Moreover, for matrix $\boldsymbol{M}(\mathcal{F}_\pi)$ we have $\text{rank}(\boldsymbol{M}(\mathcal{F}_\pi)) = \text{rank}^{\mathbb{R}}(\boldsymbol{M}(\mathcal{F}_\pi))$, thus establishing the following bounds:*

$$\text{rank}_{CP}(\mathcal{F}_\pi) \le \text{rank}^{\mathbb{R}}_{CP}(\mathcal{F}_\pi) \le n_1 n_3 \cdot \text{rank}^{\mathbb{R}}(\boldsymbol{M}(\mathcal{F}_\pi)) = n_1 n_3 \cdot \text{rank}(\boldsymbol{M}(\mathcal{F}_\pi)) \le n_1 n_3 \cdot \text{rank}_{CP}(\mathcal{F}_\pi).$$

Proposition 3.4 can be further extended to exactly compute the CP-rank for a class of tensors.

Corollary 3.6 *Consider an even order tensor $\mathcal{F} \in \mathbb{C}^{n_1 \times n_2 \times \cdots \times n_{2d}}$ such that*

$$\mathcal{F} = A^1 \otimes A^2 \otimes \cdots \otimes A^d \text{ for some matrices } A^i \in \mathbb{C}^{n_{2i-1} \times n_{2i}}.$$

Denoting $r_i = \text{rank}(A^i)$ for $i = 1, \ldots, d$, we have that $\text{rank}_{CP}(\mathcal{F}) = \text{rank}_{M^+}(\mathcal{F}) = r_1 r_2 \cdots r_d$.

3.3 Bound for super-symmetric fourth-order tensor

Theorem 3.2 essentially states that the M-rank and the symmetric CP-rank are the same in the rank-one case. This equivalence, however, does not hold in general. In this subsection, we show that although they are not equivalent, the symmetric CP-rank of $\mathcal{F} \in \mathbb{S}^{n^{2d}}$ can be both lower and upper bounded (up to a constant factor) by the corresponding M-rank.

Before providing lower and upper bounds for the symmetric CP-rank, we need the following technical results.

Lemma 3.7 *Suppose* $\sum\limits_{i=1}^{r} A^i \otimes A^i = \mathcal{F} \in \mathbb{S}^{n^4}$ *with* $A^i = \sum\limits_{j_i=1}^{m_i} a^{j_i} \otimes a^{j_i}$ *and* $a^{j_i} \in \mathbb{C}^n$ *for* $i = 1, \ldots, r$, $j_i = 1, \ldots, m_i$, $m_i \leq n$. *Then it holds that*

$$\begin{aligned} \mathcal{F} \;\; = \;\; & \sum_{i=1}^{r} \sum_{j_i=1}^{m_i} a^{j_i} \otimes a^{j_i} \otimes a^{j_i} \otimes a^{j_i} + \\ & \sum_{i=1}^{r} \sum_{j_i \neq k_i} \tfrac{1}{3} \left(a^{j_i} \otimes a^{j_i} \otimes a^{k_i} \otimes a^{k_i} + a^{j_i} \otimes a^{k_i} \otimes a^{j_i} \otimes a^{k_i} + a^{j_i} \otimes a^{k_i} \otimes a^{k_i} \otimes a^{j_i} \right). \end{aligned}$$

Proof. Since $\mathcal{F}$ is super-symmetric, we have $\mathcal{F}_{ijk\ell} = \mathcal{F}_{ikj\ell} = \mathcal{F}_{i\ell kj}$. Consequently,

$$\begin{aligned} \mathcal{F} &= \sum_{i=1}^{r} A^i \otimes A^i \\ &= \sum_{i=1}^{r} \left(\sum_{j_i=1}^{m_i} a^{j_i} \otimes a^{j_i} \right) \otimes \left(\sum_{j_i=1}^{m_i} a^{j_i} \otimes a^{j_i} \right) \\ &= \sum_{i=1}^{r} \left(\sum_{j_i=1}^{m_i} a^{j_i} \otimes a^{j_i} \otimes a^{j_i} \otimes a^{j_i} + \sum_{j_i \neq k_i} a^{j_i} \otimes a^{j_i} \otimes a^{k_i} \otimes a^{k_i} \right) & (26) \\ &= \sum_{i=1}^{r} \left(\sum_{j_i=1}^{m_i} a^{j_i} \otimes a^{j_i} \otimes a^{j_i} \otimes a^{j_i} + \sum_{j_i \neq k_i} a^{j_i} \otimes a^{k_i} \otimes a^{j_i} \otimes a^{k_i} \right) & (27) \\ &= \sum_{i=1}^{r} \left(\sum_{j_i=1}^{m_i} a^{j_i} \otimes a^{j_i} \otimes a^{j_i} \otimes a^{j_i} + \sum_{j_i \neq k_i} a^{j_i} \otimes a^{k_i} \otimes a^{k_i} \otimes a^{j_i} \right). & (28) \end{aligned}$$

The conclusion follows by taking the average of (26), (27) and (28). □

Lemma 3.8 *Suppose* $a_1, \cdots, a_m$ *are* m *vectors and* $\xi_1, \cdots, \xi_m$ *are i.i.d. symmetric Bernoulli random variables, i.e.,*

$$\xi_i = \begin{cases} -1, & \text{with probability } 1/2 \\ -1, & \text{with probability } 1/2 \end{cases} \qquad i = 1, \ldots, m.$$

Then it holds that

$$\mathsf{E}\left[\left(\sum_{j=1}^{m} \xi_j a^j \right) \otimes \left(\sum_{j=1}^{m} \xi_j a^j \right) \otimes \left(\sum_{j=1}^{m} \xi_j a^j \right) \otimes \left(\sum_{j=1}^{m} \xi_j a^j \right) \right] \tag{29}$$

$$= \sum_{j=1}^{m} a^j \otimes a^j \otimes a^j \otimes a^j + \sum_{i \neq j} \left(a^i \otimes a^i \otimes a^j \otimes a^j + a^i \otimes a^j \otimes a^i \otimes a^j + a^i \otimes a^j \otimes a^j \otimes a^i \right).$$

Proof. The expectation can be rewritten as

$$
\begin{aligned}
& \mathsf{E}\left[\left(\sum_{j=1}^m \xi_j a^j\right) \otimes \left(\sum_{j=1}^m \xi_j a^j\right) \otimes \left(\sum_{j=1}^m \xi_j a^j\right) \otimes \left(\sum_{j=1}^m \xi_j a^j\right)\right] \\
= & \sum_{i,j,k,\ell} \mathsf{E}\left[\xi_i a^i \otimes \xi_j a^j \otimes \xi_k a^k \otimes \xi_\ell a^\ell\right] \\
= & \sum_{i,j,k,\ell} \mathsf{E}\left[\xi_i \xi_j \xi_k \xi_\ell\right] a^i \otimes a^j \otimes a^k \otimes a^\ell. \qquad (30)
\end{aligned}
$$

Since $\xi_1, \cdots, \xi_m$ are i.i.d. with $\mathsf{E}[\xi_i] = 0$, $i = 1, \ldots, m$, it follows that

$$
\mathsf{E}\left[\xi_i \xi_j \xi_k \xi_\ell\right] = \begin{cases} 1, & \text{if } \{i,j,k,\ell\} = \{u,u,v,v\}, \text{ or } \{u,v,u,v\}, \text{ or } \{u,v,v,u\}, \text{ for some } u, v; \\ 0, & \text{otherwise .} \end{cases} \qquad (31)
$$

Therefore,

$$
\begin{aligned}
& \sum_{i,j,k,\ell} \mathsf{E}\left[\xi_i \xi_j \xi_k \xi_\ell\right] a^i \otimes a^j \otimes a^k \otimes a^\ell \\
= & \sum_{j=1}^m a^j \otimes a^j \otimes a^j \otimes a^j + \sum_{i \neq j} \left(a^i \otimes a^i \otimes a^j \otimes a^j + a^i \otimes a^j \otimes a^i \otimes a^j + a^i \otimes a^j \otimes a^j \otimes a^i\right),
\end{aligned}
$$

which combining with (30) yields (29). $\square$

Now we are ready to present the main result of this subsection.

Theorem 3.9 *For any given $\mathcal{F} \in \mathbb{S}^{n^4}$, it holds that*

$$
\operatorname{rank}_M(\mathcal{F}) \leq \operatorname{rank}_{SCP}(\mathcal{F}) \leq (n + 4n^2) \operatorname{rank}_M(\mathcal{F}).
$$

Proof. Let us first prove $\operatorname{rank}_M(\mathcal{F}) \leq \operatorname{rank}_{SCP}(\mathcal{F})$. Suppose $\operatorname{rank}_{SCP}(\mathcal{F}) = r$, i.e.,

$$
\mathcal{F} = \sum_{i=1}^r a^i \otimes a^i \otimes a^i \otimes a^i \text{ with } a^i \in \mathbb{C}^n \text{ for } i = 1, \ldots, r.
$$

By letting $A^i = a^i \otimes a^i$, we get $\mathcal{F} = \sum_{i=1}^r A^i \otimes A^i$ with $A^i \in \mathbb{S}^{n^2}$. Thus $\operatorname{rank}_M(\mathcal{F}) \leq r = \operatorname{rank}_{SCP}(\mathcal{F})$.

We now prove $\operatorname{rank}_{SCP}(\mathcal{F}) \leq (n + 4n^2) \operatorname{rank}_M(\mathcal{F})$. Suppose that $\operatorname{rank}_M(\mathcal{F}) = r$, then from (2.3) it holds that $\mathcal{F} = \sum\limits_{i=1}^r A^i \otimes A^i$ with $A^i \in \mathbb{S}^{n^2}$. Assume $A^i = \sum\limits_{j=1}^{m_i} a^{i_j} \otimes a^{i_j}$, $m_i \leq n$, $i = 1, \ldots, r$. Let $\xi_{1_1}, \cdots, \xi_{1_{m_1}}, \xi_{2_1}, \cdots, \xi_{r_{m_r}}$ be i.i.d. symmetric Bernoulli random variables. Then by combining

Lemmas 3.7 and 3.8, we have

$$\begin{aligned}\mathcal{F} &= \sum_{i=1}^{r} A^i \otimes A^i = \frac{2}{3} \sum_{i=1}^{r}\sum_{j=1}^{m_i} a^{i_j} \otimes a^{i_j} \otimes a^{i_j} \otimes a^{i_j} + \\ &\qquad \frac{1}{3}\sum_{i=1}^{r} \mathsf{E}\left[\left(\sum_{j=1}^{m_i} \xi_{i_j} a^{i_j}\right) \otimes \left(\sum_{j=1}^{m_i} \xi_{i_j} a^{i_j}\right) \otimes \left(\sum_{j=1}^{m_i} \xi_{i_j} a^{i_j}\right) \otimes \left(\sum_{j=1}^{m_i} \xi_{i_j} a^{i_j}\right)\right]. \end{aligned} \tag{32}$$

Assuming $\eta_{i_1}, \ldots, \eta_{i_{m_i}}$ are 4-wise independent (see, e.g., [1, 19]) and identical symmetric Bernoulli random variables for all $i = 1, \ldots, r$, we have

$$\begin{aligned} & \mathsf{E}\left[\left(\sum_{j=1}^{m_i} \xi_{i_j} a^{i_j}\right) \otimes \left(\sum_{j=1}^{m_i} \xi_{i_j} a^{i_j}\right) \otimes \left(\sum_{j=1}^{m_i} \xi_{i_j} a^{i_j}\right) \otimes \left(\sum_{j=1}^{m_i} \xi_{i_j} a^{i_j}\right)\right] \\ = \ & \mathsf{E}\left[\left(\sum_{j=1}^{m_i} \eta_{i_j} a^{i_j}\right) \otimes \left(\sum_{j=1}^{m_i} \eta_{i_j} a^{i_j}\right) \otimes \left(\sum_{j=1}^{m_i} \eta_{i_j} a^{i_j}\right) \otimes \left(\sum_{j=1}^{m_i} \eta_{i_j} a^{i_j}\right)\right], \text{ for } i = 1, \ldots, r. \end{aligned} \tag{33}$$

According to Proposition 6.5 in [1], we know that the right hand side of (33) can be written as the sum of at most $4m_i^2$ symmetric rank-one tensors. By combining (32) and (33) we further get

$$\begin{aligned}\mathcal{F} &= \frac{2}{3} \sum_{i=1}^{r}\sum_{j=1}^{m_i} a^{i_j} \otimes a^{i_j} \otimes a^{i_j} \otimes a^{i_j} + \\ &\qquad \frac{1}{3}\sum_{i=1}^{r} \mathsf{E}\left[\left(\sum_{j=1}^{m_i} \eta_{i_j} a^{i_j}\right) \otimes \left(\sum_{j=1}^{m_i} \eta_{i_j} a^{i_j}\right) \otimes \left(\sum_{j=1}^{m_i} \eta_{i_j} a^{i_j}\right) \otimes \left(\sum_{j=1}^{m_i} \eta_{i_j} a^{i_j}\right)\right], \end{aligned}$$

which leads to

$$\mathrm{rank}_{SCP}(\mathcal{F}) \le \sum_{i=1}^{r} m_i + \sum_{i=1}^{r} 4m_i^2 \le rn + 4rn^2 = (n + 4n^2)\mathrm{rank}_M(\mathcal{F}).$$

□

4 The low-M-rank tensor recovery

In this section, we consider the low-rank tensor recovery problems (2) and (4) with an emphasis on the fourth-order tensor case. In the context of tensor recovery, M^+-rank takes a *conservative* attitude towards estimating the CP-rank while M^--rank is a more *optimistic* estimation. On the middle ground, one may choose to work with a pre-specified π. In our numerical experiments, for simplicity, for a fourth-order tensor $\mathcal{F}$, we always choose to group the first two indices as the row index, and group the last two indices as the column index for square unfolding (we use $\boldsymbol{M}(\mathcal{F})$ to

denote the corresponding matrix). According to Theorems 3.3 and 3.9, by multiplying a constant factor, $\text{rank}(\boldsymbol{M}(\mathcal{F}))$ can provide an upper bound for the CP-rank and the symmetric CP-rank of $\mathcal{F}$ (if it is also super-symmetric). We denote $X = \boldsymbol{M}(\mathcal{X})$, $Y = \boldsymbol{M}(\mathcal{Y})$ and $F = \boldsymbol{M}(\mathcal{F})$. Without loss of generality, we replace the CP-rank in the objective of (2) and (4) by $\text{rank}(X)$, and it follows from Theorem 3.3 that by minimizing $\text{rank}(X)$, $\text{rank}_{CP}(\mathcal{X})$ will be small as well. In other words, rather than solving (2) and (4), we solve the following two matrix problems

$$\min_X \ \text{rank}(X), \quad \text{s.t.}, \quad \bar{\boldsymbol{L}}(X) = b, \tag{34}$$

and

$$\min_{Y,Z} \ \text{rank}(Y) + \lambda \|Z\|_0, \quad \text{s.t.}, \quad Y + Z = F, \tag{35}$$

where $\bar{\boldsymbol{L}}$ is a linear mapping such that $\bar{\boldsymbol{L}}(X) = \boldsymbol{L}(\mathcal{X})$.

It is now very natural to consider the convex relaxations of the two matrix problems (34) and (35); i.e., we replace the rank function by the nuclear norm and replace the cardinality function by the ℓ_1 norm. This results in the following two convex relaxations for (34) and (35):

$$\min_X \ \|X\|_*, \quad \text{s.t.}, \quad \bar{\boldsymbol{L}}(X) = b, \tag{36}$$

and

$$\min_{Y,Z} \ \|Y\|_* + \lambda \|Z\|_1, \quad \text{s.t.}, \quad Y + Z = F. \tag{37}$$

Note that all the variables in (34) and (35) are complex-valued. Thus, the ℓ_1 norm is defined as $\|Z\|_1 := \sum_{ij} \sqrt{(\text{Re } (Z_{ij}))^2 + (\text{Im } (Z_{ij}))^2}$. Although (34) and (35) are complex-valued problems, they can still be solved by the methods in [5, 33, 32, 2, 39] with minor modifications. We omit the details for succinctness.

When the tensors are super-symmetric, we can impose the super-symmetric constraint and get the following formulation:

$$\begin{array}{ll} \min\limits_X & \|X\|_* \\ \text{s.t.} & \bar{\boldsymbol{L}}(Y) = b, \ \boldsymbol{M}^{-1}(X) \in \mathbb{S}^{n4}, \end{array}$$

where $\boldsymbol{M}^{-1}(X)$ is the tensor whose square unfolding is X. Note that the above problem is equivalent to

$$\begin{array}{ll} \min\limits_X & \|X\|_* \\ \text{s.t.} & \boldsymbol{M}^{-1}(Y) \in \mathbb{S}^{n4}, \\ & \bar{\boldsymbol{L}}(Y) = b, \quad X = Y, \end{array} \tag{38}$$

which can be efficiently solved by the standard alternating direction method of multipliers (see the survey paper [4] for more details).

5 Numerical results

In this section, we shall show the capability of our approach in estimating the CP-rank and solving the low-CP-rank tensor recovery problems via numerical experiments.

5.1 Approximating the CP-rank via the M-rank

In this subsection, we consider the problem of estimating the CP-rank for some structured tensors. We first construct tensors in the form of

$$\mathcal{T} = \sum_{i=1}^{r} a^i \otimes b^i \otimes c^i \otimes d^i \in \mathbb{C}^{n_1 \times n_2 \times n_3 \times n_4}, \tag{39}$$

where $a^i, b^i, c^i, d^i, i = 1, \ldots, r$ are randomly generated. Obviously, the CP-rank of the resulting tensor $\mathcal{T}$ is less than or equal to r. For each set of (n_1, n_2, n_3, n_4) and r, we randomly generate 20 tensors according to (39). Then we compute the Tucker rank, the M^+-rank and the M^--rank of each tensor, and report their average value over the 20 instances in Table 1.

CP-rank	Tucker rank	M^+-rank	M^--rank
Dimension $10 \times 10 \times 10 \times 10$			
$r = 12$, CP-rank ≤ 12	(10,10,10,10)	12	12
Dimension $10 \times 10 \times 15 \times 15$			
$r = 12$, CP-rank ≤ 12	(10,10,12,12)	12	12
Dimension $15 \times 15 \times 15 \times 15$			
$r = 18$, CP-rank ≤ 18	(15,15,15,15)	18	18
Dimension $15 \times 15 \times 20 \times 20$			
$r = 18$, CP-rank ≤ 18	(15,15,18,18)	18	18
Dimension $20 \times 20 \times 20 \times 20$			
$r = 30$, CP-rank ≤ 30	(20,20,20,20)	30	30
Dimension $20 \times 20 \times 25 \times 25$			
$r = 30$, CP-rank ≤ 30	(20,20,25,25)	30	30
Dimension $25 \times 25 \times 30 \times 30$			
$r = 40$, CP-rank ≤ 40	(25,25,30,30)	40	40
Dimension $25 \times 25 \times 30 \times 30$			
$r = 40$, CP-rank ≤ 40	(30,30,30,30)	40	40

Table 1: CP-rank approximation: M-rank vs Tucker rank for tensors in (39)

From Table 1, we can see that for all instances $\mathrm{rank}_{M^+}(\mathcal{T}) = \mathrm{rank}_{M^-}(\mathcal{T}) = r$. Thus by Theorem 3.3, we can conclude that the CP-rank of these tensors is exactly r and the M-rank equals the CP-rank for these random instances. Moreover, since r is chosen to be larger than one dimension of the tensor, some components of the Tucker rank are strictly less than r.

In another setting, we generate tensors in the following manner

$$\mathcal{T} = \sum_{i=1}^{r} A^i \otimes B^i, \tag{40}$$

where matrices $A^i, B^i, i = 1, \ldots, r$ are randomly generated in such a way that $\mathrm{rank}(A^i) = \mathrm{rank}(B^i) = k, i = 1, \ldots, r$. Consequently, rk^2 is an upper bound for the CP-rank of $\mathcal{T}$. From Proposition 3.4, we know that $\mathrm{rank}_{M^+}(\mathcal{T}) = \mathrm{rank}_{CP}(\mathcal{T}) = k^2$ when $r = 1$. One may wonder if $\mathrm{rank}_{M^+}(\mathcal{T}) = \mathrm{rank}_{CP}(\mathcal{T})$ when $r > 1$. To this end, we let $r = k = 2, 3, 4, 5$ and generate 20 random instances for different choice of r, k and tensor dimensions. For each instance, we compute its Tucker rank, the M$^+$-rank and the M$^-$-rank, and report the average value over these 20 instances in Table 2.

CP-rank	Tucker rank	M$^+$-rank	M$^-$-rank
Dimension $10 \times 10 \times 10 \times 10$			
(*) $r = k = 2$, CP-rank ≤ 8	(4,4,4,4)	8	2
$r = k = 3$, CP-rank ≤ 27	(9,9,9,9)	27	3
$r = k = 4$, CP-rank ≤ 64	(10,10,10,10)	64	4
Dimension $10 \times 10 \times 15 \times 15$			
(*) $r = k = 2$, CP-rank ≤ 8	(4,4,4,4)	8	2
$r = k = 3$, CP-rank ≤ 27	(9,9,9,9)	27	3
$r = k = 4$, CP-rank ≤ 64	(10,10,15,15)	64	4
Dimension $15 \times 15 \times 20 \times 20$			
(*) $r = k = 2$, CP-rank ≤ 8	(4,4,4,4)	8	2
$r = k = 3$, CP-rank ≤ 27	(9,9,9,9)	27	3
$r = k = 4$, CP-rank ≤ 64	(16,16,16,16)	64	4
Dimension $20 \times 20 \times 20 \times 20$			
$r = k = 3$, CP-rank ≤ 27	(9,9,9,9)	27	3
$r = k = 4$, CP-rank ≤ 64	(15,15,16,16)	64	4
$r = k = 5$, CP-rank ≤ 125	(20,20,20,20)	125	5
Dimension $20 \times 20 \times 30 \times 30$			
$r = k = 3$, CP-rank ≤ 27	(9,9,9,9)	27	3
$r = k = 4$, CP-rank ≤ 64	(16,16,16,16)	64	4
$r = k = 5$, CP-rank ≤ 125	(20,20,25,25)	125	5

Table 2: CP-rank approximation: M-rank vs Tucker rank for tensors in (40)

From Table 2 we can see that $\mathrm{rank}_{M^+}(\mathcal{T}) = rk^2$ for all instances. This further implies that the CP-rank of the generated tensors is always equal to rk^2 and the M$^+$-rank equals the CP-rank. Moreover, as shown in the rows that are marked by (*) in Table 2, the CP-rank is strictly less than any dimension of the tensor, but the Tucker rank is still strictly less than the CP-rank. This again confirms that the M$^+$-rank is a much better approximation of the CP-rank compared with the averaged Tucker rank.

To summarize, results in both Table 1 and Table 2 suggest that $\text{rank}_{M+}(\mathcal{T}) = \text{rank}_{CP}(\mathcal{T})$ when $\mathcal{T}$ is randomly generated from either (39) or (40), while there is a substantial gap between the averaged Tucker rank and the CP-rank. Therefore, rank_{M+} is a better estimation than the Tucker rank for estimating the CP-rank, at least under the settings considered in our experiments.

5.2 Synthetic data for low rank tensor optimization problems

5.2.1 Low-rank tensor completion problems

In this subsection we use the FPCA algorithm proposed in [33] to solve (36) for complex-valued fourth-order tensor completion and make comparison to the low-n-rank completion approach. The testing examples are generated as follows. We generate random complex-valued tensor $\mathcal{X}_0$ based on (39) with various tensor dimensions and different value of r so that the CP-rank of the resulting tensor is less than or equal to r. Under each setting, we generate 20 instances and randomly select 70%, 50%, 30% of the entries as the observed ones for tensor completion. We also use the code in [14] to solve the low-n-rank tensor completion problem (6). We report the average of the Tucker rank, the average of the M^+-rank and the average of the M^--rank of the recovered tensor $\mathcal{X}^*$ over the 20 instances in Table 3. We also report the average of the relative errors for both approaches in Table 3, where the relative error is defined as

$$RelErr := \frac{\|\mathcal{X}^* - \mathcal{X}_0\|_F}{\|\mathcal{X}_0\|_F}.$$

The CPU times reported are in seconds. From Table 3 we can see that when the CP-rank is not very small, the low-n-rank completion approach fails to recover the original tensor while our low-M-rank method works well and its relative error is usually at the order of 10^{-5} or 10^{-6}. This result is not surprising. To illustrate this, let us take a $10 \times 10 \times 10 \times 10$ tensor as an example. In this case the mode-n unfolding and the square unfolding will result in a 10×1000 and a 100×100 matrices respectively. When the CP-rank of the underlying tensor is equal to 6, it is a relatively high rank for a 10×1000 matrix, while it is a relatively low rank for a 100×100 matrix. This is exactly what happens in the third row-block of Table 3 when the dimension of the tensor is $10 \times 10 \times 10 \times 10$ and the CP-rank is 6. In addition, we note that the Tucker rank is often larger than the CP-rank when it fails to complete the original tensor. However, the M^+-rank and M^--rank are almost always equal to the CP-rank except for only two cases. This again suggests that the M-rank is a good approximation to the CP-rank. Moreover, the results in Table 3 also show that the low-M-rank tensor completion model has a much better recoverability than the low-n-rank model in terms of the relative error of the recovered tensors. Furthermore, we conduct similar tests for our low-M-rank model (36) when the CP-rank is larger than the length of one dimension of the tensor, and only 30% of entries are observed. The averaged results over 20 randomly generated instances for

different tensor dimensions are summarized in Table 4, which shows that our low-M-rank tensor completion model can still recovery the tensor very well.

In another set of tests, we aim to observe the relationship between the M-rank and the symmetric CP-rank via solving the super-symmetric tensor completion problem (38). To this end, we randomly generate 20 complex-valued tensors $\mathcal{X}_0$ in the form (39) for different choice of tensor dimensions and values of r, so that the symmetric CP-rank of the resulting tensor is less than or equal to r. For each generated tensor, we randomly remove 60% of the entries and then solve the tensor completion problem (38). The results are reported in Table 5, which suggests that the original tensor is nicely recovered (usually with the relative error at the order of 10^{-6}). Moreover, the M-rank and the symmetric CP-rank shown in the table are *always* identical, which implies that the M-rank remains a good approximation to the symmetric CP-rank for super-symmetric tensor. We also note that solving problem (38) is much more time consuming than solving (36), due to the super-symmetric constraint which is essentially equivalent to $O(n^4)$ linear constraints and is costly to deal with.

5.2.2 Robust tensor recovery problem

In this subsection we report the numerical results for the robust tensor recovery problem (4). We choose $\lambda = 1/\sqrt{n_1 n_2}$ in (37), and apply the alternating linearization method in [2] to solve (37). The low-rank tensor $\mathcal{Y}_0$ is randomly generated according to formula (39), and a random complex-valued sparse tensor $\mathcal{Z}_0$ is generated with cardinality of $0.05 \cdot n_1 n_2 n_3 n_4$. Then we set $\mathcal{F} = \mathcal{Y}_0 + \mathcal{Z}_0$ as the observed data in (4). We conduct our tests under various settings of tensor dimensions and CP-ranks of $\mathcal{Y}_0$. For each setting, 20 instances are randomly generated for test. We also apply the code in [14] to solve the low-n-rank robust tensor recovery problem:

$$\min_{\mathcal{Y},\mathcal{Z} \in \mathbb{C}^{n_1 \times n_2 \cdots \times n_d}} \frac{1}{d} \sum_{j=1}^{d} \|Y(j)\|_* + \lambda \|\mathcal{Z}\|_1, \quad \text{s.t.} \quad \mathcal{Y} + \mathcal{Z} = \mathcal{F}. \tag{41}$$

Suppose $\mathcal{Y}^*$ and $\mathcal{Z}^*$ are the low-rank tensor and sparse tensor returned by the algorithms. We define the relative error of the low-rank tensor as

$$RelErr_{LR} := \frac{\|\mathcal{Y}^* - \mathcal{Y}_0\|_F}{\|\mathcal{Y}_0\|_F},$$

while we also consider the relative error of the constraint violation

$$RelErr_{All} := \frac{\|\mathcal{Y}^* + \mathcal{Z}^* - \mathcal{F}\|_F}{\|\mathcal{F}\|_F}.$$

We report the average of these two relative errors, the Tucker rank and the M-rank of the recovered low-rank tensor $\mathcal{Y}^*$ over 20 instances in Table 6. Results in Table 6 suggest that in many cases, the low-n-rank robust tensor recovery model fails to extract the low-rank part while our low-M-rank

robust tensor recovery model can always recover the tensor with relative error at the order of 10^{-6} or 10^{-7}. Moreover, we observe that the M-rank of the recovered tensor $\mathcal{Y}^*$ is *always* equal to the CP-rank of the original low-rank tensor $\mathcal{Y}_0$, which is consistent with the observations in the low-rank tensor completion problems.

5.3 Colored video completion and decomposition

In this subsection, we apply the matrix completion and matrix robust PCA models (34) and (35) for the purpose of colored video completion and decomposition, which can be formulated as the fourth-order tensor recovery problems (2) and (4) respectively. A colored video file consists of n_4 frames, and each frame is an image stored in the RGB format as a $n_1 \times n_2 \times 3$ array. As a result, filling in the missing entries of the colored video and decomposing the video to static background and moving foreground can be regarded as low-rank tensor completion (2) and robust tensor recovery (4), respectively.

In our experiment for tensor completion, we chose 50 frames from a video taken in a lobby, which was introduced by Li et al. in [28]. Each frame in this video is a colored image with size $128 \times 160 \times 3$. The images in the first row of Figure 1 are three frames of the video. Basically we chose the 50 frames such that they only contain static background, and thus the $128 \times 160 \times 3 \times 50$ fourth-order tensor formed by them are expected to have low rank, because the background is almost the same in each frame. We then randomly remove 80% of the entries from the video, and the images in the second row of Figure 1 are the frames after the removal. We then apply the FPCA proposed in [33] to solve (36) with the square unfolding matrix having the size 20480×150, to complete the missing entries in the target tensor. The images in the third row of Figure 1 are the frames recovered by FPCA for solving (36). We can see that we are able to recover the video very well even though 80% entries are missing.

In our experiment for robust tensor recovery, we chose another 50 frames from the same video in [28]. These frames were chosen such that the frames contain some moving foregrounds. The task in robust tensor recovery is to decompose the given tensor into two parts: a low-rank tensor corresponding to the static background, and a sparse tensor corresponding to the moving foreground. Note that the tensor corresponding to the moving foreground is sparse because the foreground usually only occupies a small portion of the frame. Thus this decomposition can be found by solving the robust tensor recovery problem (4). Here we again apply the alternating liearization method proposed in [2] to solve (35) for the task of robust tensor recovery, where λ in (37) is chosen as $1/\sqrt{n_1 n_2}$ and n_1, n_2 are the first two dimensions of the fourth-order tensor. The decomposition results are shown in Figure 2. The images in the first row of Figure 2 are frames of the original video. The images in the second and third rows of Figure 2 are the corresponding static background and moving

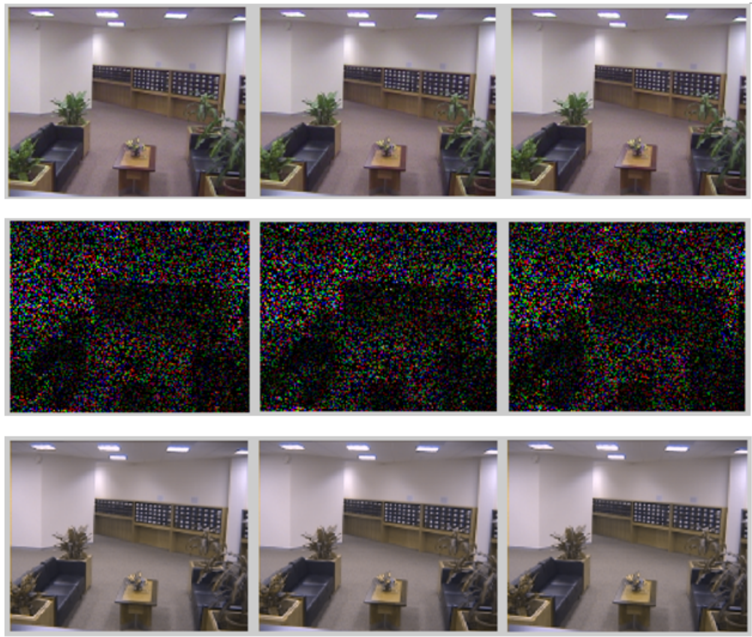

Figure 1: Results for video completion. The first row: frames of the original video; the second row: frames with 80% missing entries; the third row: recovered images using tensor completion.

foreground, respectively. We can see that our approach very effectively decomposes the video, which is a fourth-order tensor.

6 Concluding remarks

In this paper, we propose some new notions of tensor decomposition for even-order tensors, which yield some rank definitions for tensors, namely, the M-rank, the symmetric M-rank and the strongly symmetric M-rank. We show that these three definitions are equivalent if the tensor under consideration is even-order and super-symmetric. We then show that the CP-rank and symmetric CP-rank of a given fourth-order tensor can be both lower and upper bounded (up to a constant factor) by the corresponding M-rank, which provides a theoretical foundation for using the M-rank as a proximity of the CP-rank in the low-CP-rank tensor recovery problems. This is encouraging since the M-rank is much easier to compute than the CP-rank. We then solve the low-M-rank tensor recovery problems using some existing methods for matrix completion and matrix robust PCA. The results show that our method can recover the tensors very well, confirming that the M-rank is a good approximation of the CP-rank in such applications. Compared to the classical mode-n rank, the new M-rank differs in the principle to unfold the tensor into matrices: it takes

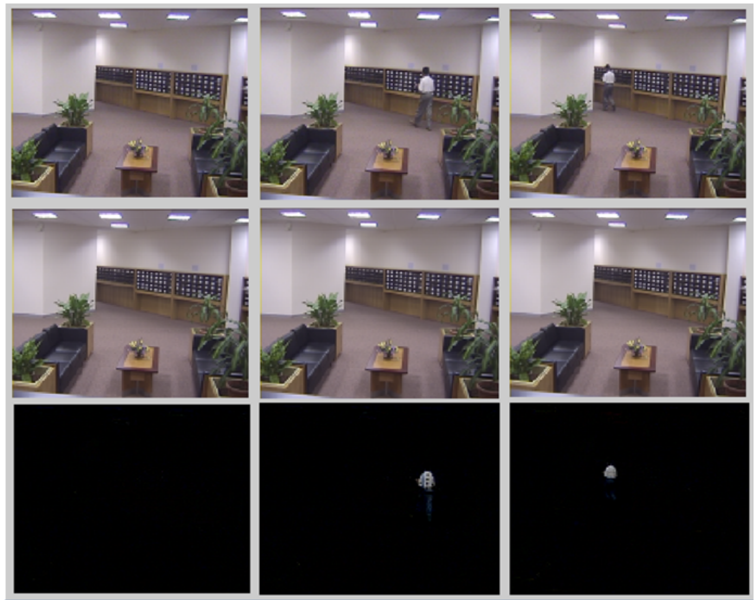

Figure 2: Robust Video Recovery. The first row are the 3 frames of the original video sequence. The second row are the recovered background, and the last row are the recovered foreground.

a balanced approach. Namely, for a $2m$-order tensor, the M-rank unfolding groups the indices by $m \times m$, while the Tucker rank folding groups the indices in the fashion of $1 \times (2m-1)$. It is certainly possibly to attempt all groupings such as $k \times (2m-k)$ with $k = 1, 2, ..., m$, though the computational costs increase exponentially with the order of the tensor. A balanced folding can also be extended to odd-order tensors; for an $(2m+1)$-order tensor, this may mean grouping the indices by $m \times (m+1)$. For the tensors of order 3, this reduces to the traditional mode-n matricization. Also, the symmetry is lost when dealing with odd-order tensors. Other than that, most results in this paper, including CP-rank approximation and tensor recovery, can be generalized straightforwardly to odd-order tensors as well.

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References

[1] N. Alon, L. Babai, and A. Itai. A fast and simple randomized algorithm for the maximal independent set problem. *Journal of Algorithms*, 7:567–583, 1986.

[2] N. S. Aybat, D. Goldfarb, and S. Ma. Efficient algorithms for robust and stable principal component pursuit. *Computational Optimization and Applications*, 58:1–29, 2014.

[3] L. Bloy and R. Verma. On computing the underlying fiber directions from the diffusion orientation distribution function. In *Medical Image Computing and Computer-Assisted Intervention, MICCAI 2008, D.Metaxas, L. Axel, G. Fichtinger and G. Szekeley, eds.*, 2008.

[4] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein. Distributed optimization and statistical learning via the alternating direction method of multipliers. *Foundations and Trends in Machine Learning*, 3(1):1–122, 2011.

[5] J. Cai, E. J. Candès, and Z. Shen. A singular value thresholding algorithm for matrix completion. *SIAM J. on Optimization*, 20(4):1956–1982, 2010.

[6] E. J. Candès and B. Recht. Exact matrix completion via convex optimization. *Foundations of Computational Mathematics*, 9:717–772, 2009.

[7] E. J. Candès and T. Tao. The power of convex relaxation: near-optimal matrix completion. *IEEE Trans. Inform. Theory*, 56(5):2053–2080, 2009.

[8] J. D. Carroll and J. J. Chang. Analysis of individual differences in multidimensional scaling via an n-way generalization of "eckart-young" decomposition. *Psychometrika*, page 283C319, 1970.

[9] P. Comon, G. Golub, L. H. Lim, and B. Mourrain. Symmetric tensors and symmetric tensor rank. *SIAM Journal on Matrix Analysis and Applications*, 30(3):1254–1279, 2008.

[10] L. De Lathauwer, B. De Moor, and J. Vandewalle. On the best rank-1 and rank-$(r_1, r_2, \ldots, r_n)$ approximation of higher-order tensors. *SIAM Journal on Matrix Analysis and Applications*, 21(4):1324–1342, 2000.

[11] M. Fazel. *Matrix Rank Minimization with Applications*. PhD thesis, Stanford University, 2002.

[12] S. Gandy, B. Recht, and I. Yamada. Tensor completion and low-n-rank tensor recovery via convex optimization. *Inverse Problems*, 27(2):025010, 2011.

[13] A. Ghosh, E. Tsigaridas, M. Descoteaux, P. Comon, B. Mourrain, and R. Deriche. A polynomial based approach to extract the maxima of an antipodally symmetric spherical function and its application to extract fiber directions from the orientation distribution function in diffusion mri. In *Computational Diffusion MRI Workshop (CDMRI08), New York*, 2008.

[14] D. Goldfarb and Z. Qin. Robust low-rank tensor recovery: Models and algorithms. *SIAM Journal on Matrix Analysis and Applications*, 35(1):225–253, 2014.

[15] R. A. Harshman. Foundations of the parafac procedure: models and conditions for an" explanatory" multimodal factor analysis. 1970.

[16] J. Håstad. Tensor rank is NP-complete. *J. Algorithms*, 11:644–654, 1990.

[17] J. J. Hilling and A. Sudbery. The geometric measure of multipartite entanglement and the singular values of a hypermatrix. *J. Math. Phys.*, 51:072102, 2010.

[18] S. Hu and L. Qi. Algebraic connectivity of an even uniform hypergraph. *Journal of Combinatorial Optimization*, 24(4):564–579, 2012.

[19] B. Jiang, S. He, Z. Li, and S. Zhang. Moments tensors, Hilbert's identity, and k-wise uncorrelated random variables. *Mathematics of Operations Research*, 39(3):775–788, 2014.

[20] B. Jiang, Z. Li, and S. Zhang. Characterizing real-valued multivariate complex polynomials and their symmetric tensor representations. *Preprint*, 2014.

[21] B. Jiang, S. Ma, and S. Zhang. Tensor principal component analysis via convex optimization. *Mathematical Programming*, 150(2):423–457, 2015.

[22] T. G. Kolda and B. W. Bader. Tensor decompositions and applications. *SIAM Review*, 51:455–500, 2009.

[23] T. G. Kolda and J. R. Mayo. Shifted power method for computing tensor eigenpairs. *SIAM J. Matrix Analysis and Applications*, 32:1095–1124, 2011.

[24] D. Kressner, M. Steinlechner, and B. Vandereycken. Low-rank tensor completion by Riemannian optimization. *BIT Numerical Mathematics*, 54(2):447–468, 2014.

[25] A. Krishnamurthy and A. Singh. Low-rank matrix and tensor completion via adaptive sampling. In *NIPS*, 2013.

[26] J. B. Kruskal. Statement of some current results about three-way arrays. *Unpublished manuscript, AT&T Bell Labs, Murray Hill, NC. Pdf available from http://three-mode.leidenuniv.nl.*, 1983.

[27] J. B. Kruskal. *Multiway Data Analysis*, chapter Rank, Decomposition, and Uniqueness for 3-way and N-way Arrays, pages 7–18. North-Holland, Amsterdam, 1989.

[28] L. Li, W. Huang, I. Gu, and Q. Tian. Statistical modeling of complex backgrounds for foreground object detection. *IEEE Trans. on Image Processing*, 13(11):1459–1472, 2004.

[29] W. Li and M. Ng. Existence and uniqueness of stationary probability vector of a transition probability tensor. Technical report, Department of Mathematics, The Hong Kong Baptist University, March 2011.

[30] L. H. Lim. Singular values and eigenvalues of tensors: a variational approach. In *Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP)*, 2005.

[31] J. Liu, P. Musialski, P. Wonka, and J. Ye. Tensor completion for estimating missing values in visual data. In *The Twelfth IEEE International Conference on Computer Vision*, 2009.

[32] Y. Liu, D. Sun, and K.-C. Toh. An implementable proximal point algorithmic framework for nuclear norm minimization. *Mathematical Programming*, 133:399–436, 2012.

[33] S. Ma, D. Goldfarb, and L. Chen. Fixed point and Bregman iterative methods for matrix rank minimization. *Mathematical Programming Series A*, 128:321–353, 2011.

[34] C. Mu, B. Huang, J. Wright, and D. Goldfarb. Square deal: Lower bounds and improved relaxations for tensor recovery. *Preprint*, 2013.

[35] L. Qi. Eigenvalues of a real supersymmetric tensor. *Journal of Symbolic Computation*, 40:1302–1324, 2005.

[36] L. Qi, G. Yu, and E. X. Wu. Higher order positive semi-definite diffusion tensor imaging. *SIAM Journal on Imaging Sciences*, pages 416–433, 2010.

[37] B. Recht, M. Fazel, and P. Parrilo. Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization. *SIAM Review*, 52(3):471–501, 2010.

[38] M. Signoretto, R. Van de Plas, B. De Moor, and J. Suykens. Tensor versus matrix completion: a comparison with application to spectral data. *IEEE Signal Processing Letters*, 18(7):403–406, 2011.

[39] M. Tao and X. Yuan. Recovering low-rank and sparse components of matrices from incomplete and noisy observations. *SIAM J. Optim.*, 21:57–81, 2011.

[40] R. Tomioka, K. Hayashi, and H. Kashima. Estimation of low-rank tensors via convex optimization. *Preprint*, 2011.

[41] R. Tomioka, T. Suzuki, and K. Hayashi. Statistical performance of convex tensor decomposition. In *NIPS*, 2011.

[42] H. Wang and N. Ahuja. Compact representation of multidimensional data using tensor rank-one decomposition. In *Proceedings of the Pattern Recognition, 17th International Conference on ICPR*, 2004.

[43] Y. Yang, Y. Feng, Huang X., and Suykens J.A.K. Rank-one tensor properties with applications to a class of tensor optimization problems. *Preprint*, 2014.

[44] X. Zhang, Z. Huang, and L. Qi. The rank decomposition and the symmetric rank decomposition of a symmetric tensor. *Prensentation in SIAM Conference on Optimization*, 2014.

sampling ratio %	low-n-rank completion		low-M-rank completion		
	Rel Err	Tucker rank	Rel Err	M^+-rank	M^--rank
Dimension $10 \times 10 \times 10 \times 10$, $r = 2$, CP-rank ≤ 2					
70%	1.23e-005	(2, 2, 2, 2)	1.83e-005	2	2
50%	1.60e-005	(2, 2, 2, 2)	1.13e-005	2	2
30%	7.79e-002	(10, 10, 10, 10)	1.18e-004	2.67	2
Dimension $10 \times 10 \times 10 \times 10$, $r = 4$, CP-rank ≤ 4					
70%	1.80e-005	(4, 4, 4, 4)	3.24e-005	4	4
50%	2.04e-001	(10, 10, 10, 10)	1.78e-005	4	4
30%	6.31e-001	(10, 10, 10, 10)	2.69e-004	5.33	4
Dimension $10 \times 10 \times 10 \times 10$, $r = 6$, CP-rank ≤ 6					
70%	1.69e-001	(10, 10, 10, 10)	4.69e-005	6	6
50%	4.50e-001	(10, 10, 10, 10)	2.62e-005	6	6
30%	7.18e-001	(10, 10, 10, 10)	4.97e-006	6	6
Dimension $15 \times 15 \times 15 \times 15$, $r = 3$, CP-rank ≤ 3					
70%	1.24e-005	(3, 3, 3, 3)	6.67e-006	3	3
50%	1.63e-005	(3, 3, 3, 3)	1.83e-006	3	3
30%	1.57e-003	(3, 3, 3, 3)	5.49e-005	3	3
Dimension $15 \times 15 \times 15 \times 15$, $r = 6$, CP-rank ≤ 6					
70%	1.57e-005	(6, 6, 6, 6)	1.22e-005	6	6
50%	1.73e-001	(15, 15, 15, 15)	3.90e-006	6	6
30%	6.10e-001	(15, 15, 15, 15)	7.78e-005	6	6
Dimension $15 \times 15 \times 15 \times 15$, $r = 9$, CP-rank ≤ 9					
70%	1.56e-001	(15, 15, 15, 15)	2.13e-005	9	9
50%	4.47e-001	(15, 15, 15, 15)	6.80e-006	9	9
30%	7.16e-001	(15, 15, 15, 15)	1.40e-004	9	9
Dimension $20 \times 20 \times 20 \times 20$, $r = 4$, CP-rank ≤ 4					
70%	7.26e-006	(4, 4, 4, 4)	3.49e-006	4	4
50%	1.45e-005	(4, 4, 4, 4)	6.76e-007	4	4
30%	1.96e-005	(4, 4, 4, 4)	2.37e-005	4	4
Dimension $20 \times 20 \times 20 \times 20$, $r = 8$, CP-rank ≤ 8					
70%	1.67e-005	(8, 8, 8, 8)	6.60e-006	8	8
50%	1.34e-001	(20, 20, 20, 20)	1.59e-006	8	8
30%	5.87e-001	(20, 20, 20, 20)	4.24e-005	8	8
Dimension $20 \times 20 \times 20 \times 20$, $r = 12$, CP-rank ≤ 12					
70%	1.45e-001	(20, 20, 20, 20)	8.58e-006	12	12
50%	4.20e-001	(20, 20, 20, 20)	3.46e-006	12	12
30%	6.99e-001	(20, 20, 20, 20)	6.75e-005	12	12

Table 3: Low-M-rank tensor completion vs. low-n-rank tensor completion

$\text{rank}_{CP}(\mathcal{X}_0)$	$RelErr$	CPU	$\text{rank}_{M^+}(\mathcal{X}^*)$	$\text{rank}_{M^-}(\mathcal{X}^*)$
Dimension $10 \times 10 \times 10 \times 10$				
$r = 12,\ \text{rank}_{CP}(\mathcal{X}_0) \leq 12$	1.47e-005	13.9	12	12
Dimension $10 \times 10 \times 12 \times 12$				
$r = 15,\ \text{rank}_{CP}(\mathcal{X}_0) \leq 15$	2.12e-005	22.29	15	15
Dimension $20 \times 20 \times 20 \times 20$				
$r = 24,\ \text{rank}_{CP}(\mathcal{X}_0) \leq 24$	2.96e-006	267.91	24	24
Dimension $20 \times 20 \times 25 \times 25$				
$r = 28,\ \text{rank}_{CP}(\mathcal{X}_0) \leq 28$	1.73e-004	11.26	28	28
Dimension $30 \times 30 \times 30 \times 30$				
$r = 35,\ \text{rank}_{CP}(\mathcal{X}_0) \leq 35$	6.63e-005	56.27	35	35
Dimension $35 \times 35 \times 40 \times 40$				
$r = 45,\ \text{rank}_{CP}(\mathcal{X}_0) \leq 45$	4.05e-005	138.77	45	45

Table 4: Numerical results for low-CP-rank tensor completion by solving (36)

$\text{rank}_{SCP}(\mathcal{X}_0)$	$RelErr$	CPU	$\text{rank}_M(\mathcal{X}^*)$
Dimension $10 \times 10 \times 10 \times 10$			
$r = 8,\ \text{rank}_{SCP}(\mathcal{X}_0) \leq 8$	8.83e-006	5.96	8
$r = 12,\ \text{rank}_{SCP}(\mathcal{X}_0) \leq 12$	6.81e-006	8.33	12
Dimension $15 \times 15 \times 15 \times 15$			
$r = 8,\ \text{rank}_{SCP}(\mathcal{X}_0) \leq 8$	8.95e-006	64.66	8
$r = 20,\ \text{rank}_{SCP}(\mathcal{X}_0) \leq 20$	6.19e-006	89.40	20
Dimension $20 \times 20 \times 20 \times 20$			
$r = 15,\ \text{rank}_{SCP}(\mathcal{X}_0) \leq 15$	6.63e-006	523.54	15
$r = 25,\ \text{rank}_{SCP}(\mathcal{X}_0) \leq 25$	6.46e-006	567.30	25
Dimension $25 \times 25 \times 25 \times 25$			
$r = 15,\ \text{rank}_{SCP}(\mathcal{X}_0) \leq 15$	4.21e-006	1109.26	15
$r = 30,\ \text{rank}_{SCP}(\mathcal{X}_0) \leq 30$	3.87e-006	2470.67	30

Table 5: Numerical results for low-CP-rank super-symmetric tensor completion by solving (38)

	low-n-rank robust tensor recovery			low-M-rank robust tensor recovery			
$\text{rank}_{CP}(\mathcal{Y}_0)$	Rel Err_{All}	Rel Err_{LR}	Tucker rank	Rel Err_{All}	Rel Err_{LR}	$\text{rank}_{M+}(\mathcal{Y}_0)$	$\text{rank}_{M-}(\mathcal{Y}_0)$
Dimension $10 \times 10 \times 10 \times 10$							
CP-rank ≤ 2	1.69e-004	1.51e-002	(3.85, 3.80, 3.90, 3.85)	8.48e-007	2.97e-007	2	2
CP-rank ≤ 4	2.77e-004	1.85e-002	(5.30, 5.35, 5.15, 5.20)	7.76e-007	5.93e-007	4	4
CP-rank ≤ 6	4.10e-004	2.19e-002	(10, 10, 10, 10)	8.63e-007	1.86e-006	6	6
CP-rank ≤ 8	4.70e-004	2.35e-002	(10, 10, 10, 10)	7.78e-007	2.44e-006	8	8
CP-rank ≤ 12	4.96e-004	2.41e-002	(10, 10, 10, 10)	8.61e-007	6.75e-006	12	12
Dimension $15 \times 15 \times 15 \times 15$							
CP-rank ≤ 3	6.77e-005	2.88e-003	(5.15, 5.15, 5.25, 5.25)	8.50e-007	2.31e-007	3	3
CP-rank ≤ 6	4.28e-004	7.00e-003	(15, 15, 15, 15)	8.57e-007	3.08e-007	6	6
CP-rank ≤ 9	4.38e-004	1.12e-002	(15, 15, 15, 15)	8.13e-007	3.33e-007	9	9
CP-rank ≤ 12	4.68e-004	1.29e-002	(15, 15, 15, 15)	7.54e-007	3.31e-007	12	12
CP-rank ≤ 18	4.97e-004	1.22e-002	(15, 15, 15, 15)	8.11e-007	4.04e-007	18	18
Dimension $20 \times 20 \times 20 \times 20$							
CP-rank ≤ 4	4.82e-004	8.05e-003	(20, 20, 20, 20)	8.64e-007	1.87e-007	4	4
CP-rank ≤ 8	4.68e-004	9.91e-003	(20, 20, 20, 20)	7.37e-007	2.36e-007	8	8
CP-rank ≤ 12	4.53e-004	1.07e-002	(20, 20, 20, 20)	8.59e-007	3.09e-007	12	12
CP-rank ≤ 16	4.69e-004	1.09e-002	(20, 20, 20, 20)	8.26e-007	3.31e-007	16	16
CP-rank ≤ 24	4.97e-004	9.42e-003	(20, 20, 20, 20)	8.59e-007	4.00e-007	24	24

Table 6: Low-M-rank robust tensor recovery vs. low-n-rank robust tensor recovery